Contents

1 Overview

Main Physical Results:

1. *At certain regions in its moduli space of vacua, (four-dimensional)* $\mathcal{N} = 2$, $SU(3)$ *Super-Yang-Mills (SYM) possesses a density of states (DOS) that grows exponentially with the mass. Moreover, this exponential growth is contributed by BPS states.*
2. *The BPS-indices (a weighted count of BPS states) in a large class of* $\mathcal{N} = 2$ *supersymmetric field theories (theories of class* $S[A_{K-1}]$*) are determined by algebraic equations. From a naïve standpoint, these algebraic equations seem to suggest that an exponential growth in the DOS is a "generic" phenomenon.*

Why is this interesting?

1. Exponential-growth in DOS in a field theory seems to violate standard thermodynamic arguments, unless the BPS states contributing to the growth are allowed to grow to arbitrarily large sizes ("BPS Giants").
2. Exponential growth of DOS $\Rightarrow$ "Hagedorn" limiting temperatures $\Rightarrow$ possible interesting phase transitions.
3. In $SU(3)$ SYM, there may be infinitely many Hagedorn temperatures that appear in the theory (corresponding to different families of BPS states, each labeled by a rational number).

2 Broken Expectations

Start with a UV Complete, field theory T_* (e.g. asymptotically free) $\rightsquigarrow$ large-energy phenomena are well-described by a Conformal Field Theory (CFT).

Now imagine a $(d-1,1)$-CFT (i.e. a d-dimensional CFT on Lorentz-signature spacetime) confined to a box of volume V and heated up to a temperature T (with respect to a particular splitting of space and time).

Via dimensional analysis

$$\text{Energy in box} = E(V,T) = \alpha V T^d$$
$$\text{Entropy} = S(T,V) = \beta V T^{d-1}$$

for some constants $\alpha, \beta > 0$. Thus,

$$S(E,V) = \kappa V^{1/d} E^{(d-1)/d}.$$

(where $\kappa = \beta\alpha^{(1-d)/d}$). But, then

$$\#\text{ of states at Energy } E \text{ in CFT} = Ce^{S(E,V)} = Ce^{\kappa V^{1/d} E^{(d-1)/d}}$$

Thus, in the theory T_*, the asymptotic growth of the number of states in the theory of mass $\leq M$ should be $c_1 \exp(c_2 E^{3/4})$ in dimension $d = 4$.

3 Motivation

Here is some (personal and probably ahistorical) motivation for why one would want to study BPS states in the first place:

1. *Caveman-question*: *What is the 1-particle spectrum of some field theory* T_**?* (Typically this is a hard question.)
2. *Sophisticated-question*: Embed T_* into a family of theories:

$$\mathsf{T}_* \rightsquigarrow \text{Family of theories } \{\mathsf{T}_u\}_{u\in\mathcal{M}}$$

   where $\mathcal{M}$ is some "moduli-space" (e.g. the moduli space of vacua). Then ask "*What is the 1-particle spectrum that is* <u>*stable*</u>[1] *as we vary* $u \in \mathcal{M}$*?*"

In the context of four-dimensional $\mathcal{N} = 2$ Field Theories: a (maybe partial) answer to the second question is the BPS spectrum.

4 Recollections on four-dimensional $\mathcal{N} = 2$ Field theory

4.1 BPS Irreps

- Four-dimensional $\mathcal{N} = 2$ Field Theories $\rightsquigarrow$ Unitary representations of the $\mathcal{N} = 2$ Super-Poincaré Algebra: 8 real odd-supercharges + a (complex-valued) central element $\widehat{Z}$ in addition to the Poincaré algebra.
- Irreps are classified by **Spin** $J^2 \in \frac{1}{2}\mathbb{Z}$, **Mass** $M^2 > 0$, *and* ***Central Charge*** $Z \in \mathbb{C}$.
- In any irrep we must have

$$M > |Z|.$$

- BPS irrep: 1-particle irrep with $|Z| = M$.

4.2 Low Energy Effective Field Theory

Typically, at low energies, the effective field theory (EFT) is an $\mathcal{N} = 2$ *abelian* gauge theory:

[1] Assuming we have some notion of parallel transport on the moduli space.

4.2.1 Data Seen by the Low-Energy EFT

1. $\mathcal{B}$: Coulomb branch (part of the space of vacua for the field theory)
2. $\widehat{\Gamma} \to \mathcal{B}$: local system of possible electric/mag./flavour charges $\widehat{\Gamma}_u$ for $u \in \mathcal{B}$ is the lattice of charges over u. (e.g. $\Gamma_u \cong \mathbb{Z} \times \mathbb{Z}$ for a pure $U(1)$ gauge theory). Each $\widehat{\Gamma}_u$ is equipped with an antisymmetric pairing $\langle \cdot, \cdot \rangle_u : \widehat{\Gamma}_u^{\otimes 2} \to \mathbb{Z}$ (the pairing between electric and magnetic charges).
3. $Z_u : \widehat{\Gamma}_u \to \mathbb{C}$ the central charge function. (From the charge in $\widehat{\Gamma}_u$, one can determine the central charge.)

4.2.2 BPS States

BPS states are massive excitations of the full theory T that remain "stable" as we vary $u \in \mathcal{B}$.

We define a counting-index (the BPS-index/Second-Helicity Supertrace/Donaldson-Thomas invariant):

$$\begin{aligned}\Omega(\gamma; u) &= \text{Weighted Count of BPS states of charge } \gamma \in \widehat{\Gamma}_u \text{ in } \mathsf{T}_u \text{ (theory with vacuum } u \in \mathcal{B}\text{)} \\ &= \text{Weighted-Super-Trace}\left[\mathcal{H}^{\text{1-part}}_{\text{BPS}}(\gamma)\right]\end{aligned}$$

- $\Omega(\gamma; u)$ is piecewise constant on $u \in \mathcal{B}$; jumps across (real-codimension 1) *walls of marginal stability* on $\mathcal{B}$.
- This jumping behaviour is well-understood via the Kontsevich-Soibelman Wall-Crossing-Formula (Kontsevich-Soibelman, Gaitto-Moore-Neitzke) $\Rightarrow$ in principle, if the location of all walls are known, then knowing $\Omega(\gamma; u_*)$ at some point $u_* \in \mathcal{B}$ determines $\Omega(\gamma; u)$ for all $u \in \mathcal{B}$.

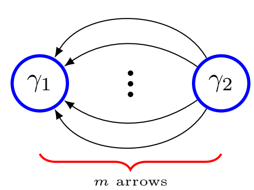
Example Seiberg/Witten: $\mathcal{N} = 2$, $\mathcal{SU}(2)$ SYM

Techniques for computing $\Omega(\gamma; u)$ at a point $u \in \mathcal{B}$:

1. BPS Quivers (requires partial knowledge of the BPS spectrum): nodes are labelled by specific collection of charges $\{\gamma_i\}_{i=1}^n \subset \widehat{\Gamma}_u$ and the number of arrows between a pair of nodes (including their direction) is specified via the pairing $\langle \cdot, \cdot \rangle_u : \widehat{\Gamma}_u^{\otimes 2} \to \mathbb{Z}$.
2. Spectral networks (Theories of Class $S[A_{K-1}]$)

5 The m-Kronecker Quiver

Suppose we know that T_u contains two so-called "hypermultiplets" of charge γ_1, $\gamma_2 \in \widehat{\Gamma}_u$ such that $\langle \gamma_1, \gamma_2 \rangle_u = m > 0$. This corresponds to the m-Kronecker quiver:



- Using quiver-technology, when $m \geq 3$ we can predict that T_u contains an extremely large family of BPS states[2] of charges $a\gamma_1 + b\gamma_2$ where a/b is a rational number on some interval: the set of phases of BPS states densely fill an arc on the circle.

[2]Technically one also needs to check that the stability parameter on the quiver, specified by the central charge, is in the "non-trivial" region of the m-Kronecker stability parameter space.

- An exponential growth of BPS indices for states of charge $n(\gamma_1+\gamma_2)$ (a corollary of the algebraic equations conjectured by Kontsevich-Soibelman and proven by Reineke):

$$\Omega(n(\gamma_1+\gamma_2)) \sim (-1)^{mn+1} K_m n^{-5/2} e^{c_m n}$$

  where

$$K_m = \frac{1}{m-1}\sqrt{\frac{m}{2\pi(m-2)}}$$
$$c_m = \log\left[(m-1)^{2(m-1)^2} m(m-2)^{m(m-2)}\right].$$

  Certain closely related results (T. Weist's asymptotics of Euler Characteristics) seem to suggest that this exponential growth will occur for most (if not all) BPS states whose phase lies on the dense arc.

This is in direct conflict with the thermodynamic prediction on asymptotics:

- The mass of a BPS state of charge $n\gamma$ is:

$$E_{n\gamma} = |Z_{n\gamma}| = |n||Z_\gamma| = |n|E_\gamma.$$

- The number of BPS states of charge $n\gamma$ is bounded below by $4|\Omega(n\gamma)|$:

$$|\Omega(n\gamma)| \leq \frac{1}{4}\{\#\text{ of BPS states of charge } n\gamma\}.$$

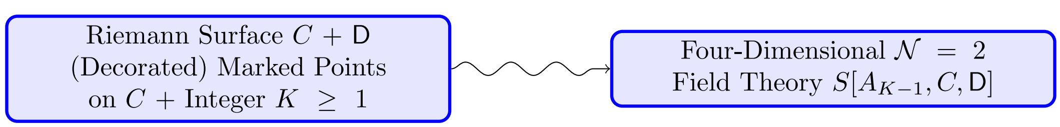
Hence the large n asymptotics of "# of BPS states of charge $n\gamma$" grows *at least as fast* as

$$K_m n^{-5/2} e^{c_m n}; \tag{1}$$

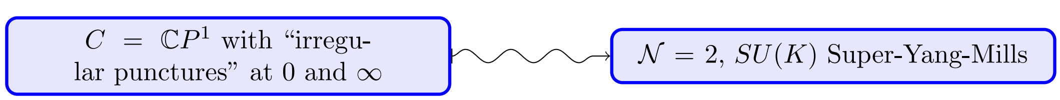
which grows strictly faster than $e^{Cn^{3/4}}$ for any constant C, i.e. the ratio of (1) to $\exp(Cn^{3/4})$ tends to infinity for any C. Thus, naïvely, the m-Kronecker quiver cannot possibly occur as a BPS (sub)-quiver.

6 Spectral Networks

6.1 Theories of class $S[A_{K-1}]$

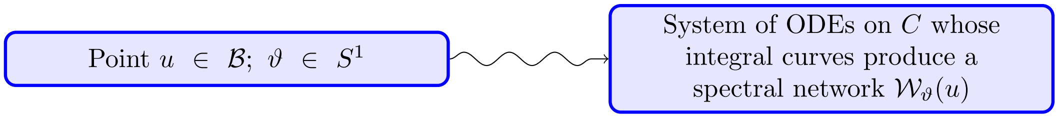


E.g,

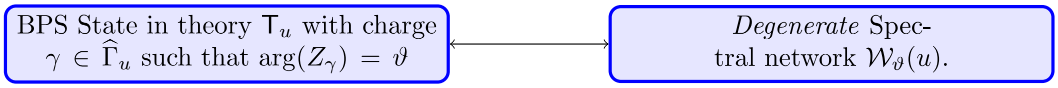


The Coulomb branch $\mathcal{B}$ of a theory of class $S[A_{K-1}]$ is a (complex) dimension $K-1$ complex vector space.

- *Spectral networks are decorated, directed graphs on C.*
- They are produced by differential equations.



- Degenerate networks – very special networks with integral curves moving in “opposite directions” that collide – correspond to BPS states:



- Off of Walls of marginal stability all BPS States of phase ϑ have proportional charges. Hence, *a degenerate network $\mathcal{W}_\vartheta(u)$ is representative of all BPS states of charge $n\gamma$ for $\gamma \in \widehat{\Gamma}_u$ a “primitive” charge with $\arg(Z_\gamma) = \vartheta$, and $n \in \mathbb{Z}$ any integer. In fact, $\mathcal{W}_\vartheta(u)$ encodes the information about all BPS indices $\Omega(n\gamma)$.*

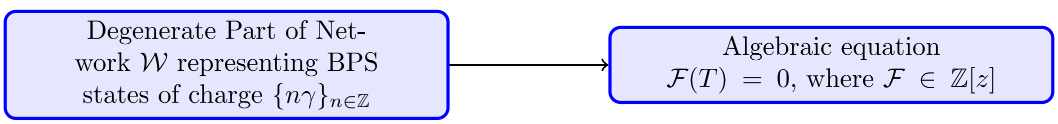
Examples of degenerate spectral networks that appear when $K = 2$:

6.2 Determining $\Omega(n\gamma)$

Central to the spectral network machinery is the generating function:

$$T_\gamma = \prod_{n=1}^{\infty} (1 - (\pm 1)^n z^n)^{n\Omega(n\gamma)}$$

Then, given the degenerate part of the network (A *finite* directed graph in all known examples), there is a combinatorial algorithm for producing an algebraic equation satisfied by T:



Examples

1. Hypermultiplet network: $T = 1 - z$, i.e.

$$\Omega(n\gamma) = \begin{cases} 1, & n = 1; \\ 0, & n > 1. \end{cases}$$

2. Vectormultiplet network: $T = (1 - z)^{-2}$, i.e.

$$\Omega(n\gamma) = \begin{cases} -2, & n = 1; \\ 0, & n > 1. \end{cases}$$

3. m-herd: $T = P^m$ where P satisfies the equation $\mathcal{F}(P) = P - zP^{(m-1)^2} - 1 = 0$. There are infinitely many non-vanishing $\Omega(n\gamma)$ corresponding to the $\Omega(n(\gamma_1 + \gamma_2))$ associated to the m-Kronecker Quiver $\Rightarrow$ exponential degeneracy.

4. $(3, 2|m)$-herd: T satisfies a complicated degree 39 polynomial equation; corresponding BPS indices are $\Omega(n(a\gamma_1 + b\gamma_2))$ and have large n asymptotics of the form $Kn^{-5/2}\exp(cn)$.

Corollary 6.1 *The generating function T_γ is an algebraic function over $\mathbb{Q} \Rightarrow$ it is a holomorphic function for z in some neighbourhood of $\mathbb{C}$; it analytically continues to a Riemann surface (which we can determine from the polynomial $\mathcal{F} \in \mathbb{Z}[z]$).*

With a bit more work (and information) one can deduce the following:

Corollary 6.2 *BPS indices coming from spectral networks (with finite degenerate parts) have asymptotics of the form*

$$\Omega(n\gamma) \sim C n^{\alpha} \sum_{i=1}^{k} \left(\frac{1}{\rho_i}\right)^n$$

where $\alpha \in \mathbb{Q}$, *and the* $\rho_i \in \mathbb{C}$ *are all algebraic numbers with the same modulus* $|\rho_i| \leq 1$.

Assuming the most generic possible polynomial $\mathcal{F}$, we expect asymptotics of the form

$$\Omega(n\gamma) \sim C n^{-5/2} \left(\frac{1}{\rho}\right)^n$$

where $\rho \in \overline{\mathbb{Q}} \cap (-1, 1)$; hence, from this naïve perspective, we should almost always expect an exponential growth of BPS states.