

RESEARCH STATEMENT

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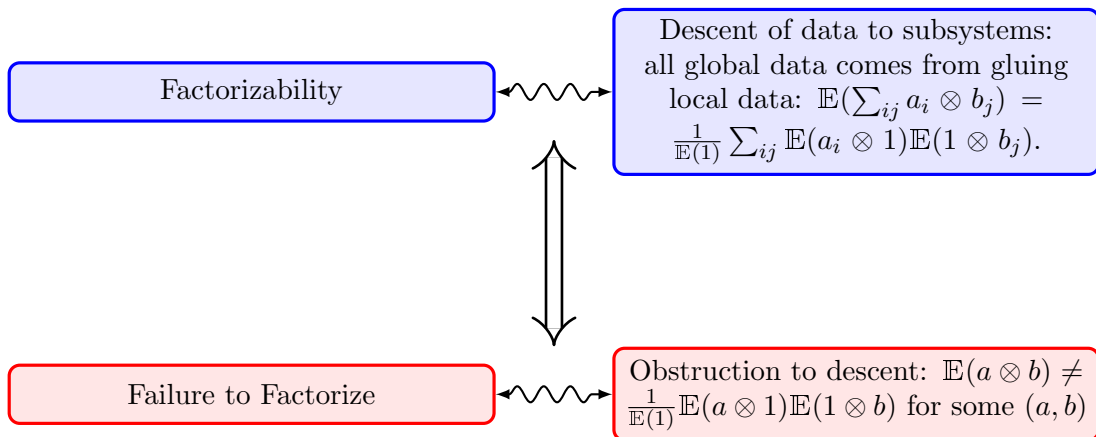
1. OVERVIEW

My interests reside near the vast range of topics at the intersection of quantum field theory/string theory, quantum information, and geometry. My recent focus is on the use of homological/homotopical techniques to understand entanglement, with an eye toward applications to quantum information and gauge/gravity duality. My time as a graduate student and early postdoc was spent thinking about four-dimensional $\mathcal{N} = 2$ supersymmetric field theories.

2. ENTANGLEMENT, COHOMOLOGY, AND THE CATEGORIFICATION OF MUTUAL INFORMATION

Given a tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, the set of entangled states—states that *fail* to factorize as $\psi_1 \otimes \cdots \otimes \psi_n$ for some $\psi_i \in \mathcal{H}_i$ —play a fundamental role in quantum information/computation [1, 2]. The study of entangled states has recently spilled over into high-energy physics, particularly after the work of Ryu-Takayanagi [3], relating entanglement entropy (a numerical measure of entanglement) of states in a conformal field theory to the area of minimal surfaces in a gravity dual theory. Some of these ideas have discrete, computationally accessible versions [4, 5, 6] using the techniques of tensor networks [7, 8, 9] and error correcting codes.

My recent focus is on a cohomological measure of entanglement: to every multipartite state there is a graded vector space—resulting from the cohomology of a cochain complex—that provides a measure of how badly a state fails to factorize. The intuition behind the existence of such cohomologies for a bipartite state $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is provided by the following diagram (where $\mathbb{E}(x) := \langle \psi, x\psi \rangle$ for $x \in \text{End}(\mathcal{H}_1 \otimes \mathcal{H}_2)$).



Cohomology is precisely formulated to detect obstructions of this form. I was able to realize the above idea in [10]: given an arbitrary multipartite state, I was able to construct cochain complexes whose associated cohomology outputs tuples of multi-body operators that are “non-locally” correlated: a phenomenon related to entangled states and made famous by accounts of the Einstein-Podolsky-Rosen paradox. This machinery, however, is not restricted purely to the quantum regime: one can also construct cochain complexes measuring how badly a probability measure on a product of measurable spaces fails to split as a product measure: i.e. the failure

of independence of commutative random variables distributed by a joint probability distribution. From the perspective of non-commutative measure theory, this classical probabilistic question, and the quantum mechanical question about factorizability of a state are special cases of a question regarding the factorizability of a multipartite positive linear functional—the “expectation value”—on W^* -algebras: special types of C^* -algebras that abstract von Neumann algebras, and play the role of algebras of non-commutative random variables. The cochain complexes alluded to above are constructed most naturally in this language.¹

Moreover, these cochain complexes can be thought of as a step toward a categorification of *multivariate mutual informations*: \mathbb{R} -valued quantities defined as an alternating sum of entropies that quantify the information shared among all subsystems [11]. The multivariate mutual information associated to a multipartite expectation value vanishes if the expectation value is factorizable, making it a good indicator of factorizability. Bivariate mutual information is particularly famous among high energy theorists, having important finiteness properties for field-theoretic states [12]. Given a multipartite expectation value, I define an associated three-parameter invariant²—called the *state index*—which behaves like an “Euler characteristic” in the sense that it behaves well under tensor products of expectation values and a certain notion of “gluing” of expectation values. Moreover, the state index is holomorphic in each of its parameters; by taking certain limits we see that it interpolates between multipartite mutual information, Tsallis/Renyí deformations [13, 14, 15, 16] of mutual information [10, §8.5], and the integer-valued Euler characteristics of our complexes.

In a more sophisticated treatment, one should be able to recover the full three-parameter state index from cohomological techniques. But even for the complexes of vector spaces developed so far, cohomology gives more than a numerical measure of how much information is shared between subsystems: elements of cohomology specify *what* information is shared among subsystems in the form of tuples of random variables. Moreover, there are entangled states for which multivariate mutual information vanishes, but have non-vanishing cohomology (see §2.3). This is analogous to how the Euler characteristic of any orientable compact manifold of odd-dimension is vanishing, making it useless for detecting the difference between two such manifolds (as opposed to the calculation of homology).

This work is closely related to that of Baudot and Bennequin [17] (J.P. Vigneaux provides an excellent exposition in [18]), who independently construct chain complexes of functions on spaces of probability measures such that mutual informations (and their Tsallis q -deformations) arise as generators of the first cohomology group. My work differs from theirs in the sense that I have a cohomology theory associated to a fixed measure (rather than the space of measures), allowing the mutual informations to arise as Euler characteristics rather than cocycles; however, the two theories are undoubtedly intimately related. I am hoping to elucidate this relationship, perhaps by carefully phrasing their construction in the language of obstruction theory and classifying stacks.

2.1. Software. As a hands-on supplement to [10], I developed software for calculating the cohomologies of multipartite quantum states, which is available on GitHub.³ At the simplest level, the software takes in an N -partite (density) state and outputs associated degree $(N - 1)$ Poincaré polynomials; however, it can also be used to compute explicit generators for cohomology.

¹Using ideas behind the Gelfand-Neumark-Segal construction of $*$ -representations of C^* -algebras, given a multipartite state one constructs a presheaf of vector spaces over the set of tensor factors; cochain complexes then arise as Čech complexes associated to this presheaf.

²Here “invariant” means that this function is invariant under automorphisms of random variables that decompose as a tensor product of automorphisms on each tensor factor.

³See: <https://github.com/tmainiero/homological-tools-4QM-octave> and <https://github.com/tmainiero/homological-tools-4QM-mathematica>

2.2. Goals and Future Work. A few interrelated goals of this research are as follows:

- (1) New invariants constructed from topological field theories: e.g. link invariants in Chern Simons theory.
- (2) Applications to the study of symmetry protected topological phases.
- (3) The development of measures of shared information both quantum mechanically, and classically: the quantum mechanical case being relevant to entanglement in quantum information theory and the latter possibly relevant to the analysis of large data sets;
- (4) An understanding of the role of homotopy theory/homological algebra in *non-commutative probability theory*: the study of expectation value functions on (possibly) non-commutative algebras;
- (5) Developing rigorous techniques for exploring the connection between entanglement and non-trivial geometries or topologies (e.g. the “gauge/gravity” or “AdS/CFT” correspondence).

With regard to (1): One can use Chern Simons theory [19, 20] to assign N -partite state to complements of (framed) N -component links in the three sphere.⁴ By taking Poincaré polynomials of the cohomologies associated to such multipartite states, one produces link invariants. The relation of these link invariants to known link invariants remains unknown.

With regard to (2): There is a generalization of these homological techniques to the situation where one has an action of some symmetry group G on each tensor factor; in particular, there is a generalization of the state index to such a G -equivariant situation—giving an interesting G -equivariant generalization of mutual information/entanglement entropy that might be suitable to the study of symmetry protected topological phases of matter. I am currently exploring such applications in the context of matrix product states (which model the ground states of gapped Hamiltonians on one-dimensional lattices [24]); this research also has deep ties to (1) (see: [25, 26]).

With regard to (4): the machinery behind these complexes suggests that multipartite states behave like spaces, and there is a whole homotopy theory associated to them. Indeed, to every multipartite expectation value, there is a geometric object: formally given by a simplicial object in an appropriate “category of states”. Our cohomological techniques are simple probes of this geometric object.

Moreover, there are certainly connections with the work of Drummond-Cole, Park, and Terilla [27, 28, 29, 30] who approach non-commutative probability theory from an A_∞/L_∞ -perspective. One sophisticated version of the chain complexes above admits the structure of a differential graded module for a differential graded algebra; the resulting cohomology is an A_∞ -module of some A_∞ -algebra. Very little is known about these higher algebraic structures at the moment but, drawing vague analogies with the way that Massey products can identify subtle linkage properties of knot complements in the three sphere, my hope is that the higher structures can detect the Borromean-like entanglement properties of the tripartite Greenberger-Horne-Zeilinger (GHZ) state [31, 32].

I also believe that a homological/homotopical approach to (non-commutative) probability theory might lead to a geometrically inspired proof of strong subadditivity: a relation between entropies on a tripartite system, expressible as a simple inequality between bipartite mutual informations. Strong subadditivity of quantum entropies is a non-trivial statement, first proven by Leib and Ruskai [33]. However, Ryu-Takayanagi formulae lead to a very simple geometric interpretation of quantum strong subadditivity; the caveat is that these formulae only hold for small class of states that have special field theoretic descriptions. The homological/homotopical techniques mentioned above are not bound to this limitation.

With regard to (5): as discussed above, there is a deep duality between the classical geometry of general relativity and entangled states: e.g. the Ryu-Takayanagi formula, or even more vague

⁴Such states are studied in [21, 22, 23].

statements such as EPR (Einstein-Podolsky-Rosen) = ER (Einstein-Rosen) [34]. Currently, numerical quantities such as mutual informations or entanglement entropies are used to study this duality; however, by their very nature as methods for studying space, cohomological techniques are likely to shed more light. Furthermore, it would be interesting if there were a connection with other perspectives on entanglement which draw on ideas of the linkage between entanglement and topology by associating entangled states to braids [35].

2.3. A Sample of Results. As a small taste of the cohomological techniques above, we begin by providing a theorem for bipartite pure states. In the following: if \mathcal{H} is a Hilbert space $\text{End}(\mathcal{H})$ denotes its space of (bounded) endomorphisms.

Theorem 2.1

Let \mathcal{H}_A and \mathcal{H}_B be Hilbert spaces and $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$. Then there is a chain complex $G(\psi)$ of \mathbb{C} -vector spaces concentrated in degrees $-1, 0$ and 1 :

$$0 \rightarrow \mathbb{C} \xrightarrow{d_{\text{aug}}} G_A^\psi \times G_B^\psi \xrightarrow{d_0} \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow 0$$

such that

- (1) $H^0[G(\psi)] := \ker(d_0)/\text{Im}(d_{\text{aug}})$ is naturally identifiable with a quotient of the space of pairs $(a, b) \in \text{End}(\mathcal{H}_A) \times \text{End}(\mathcal{H}_B)$ that are “maximally correlated” in a precise sense (in particular, $\langle \psi, (a \otimes 1_B - 1_A \otimes b)^*(a \otimes 1_B - 1_A \otimes b)\psi \rangle = 0$);
- (2) $\dim_{\mathbb{C}} H^0[G(\psi)] = r^2 - 1$, where r is the “Schmidt rank” of ψ ; in particular ψ is entangled if and only if $H^0 \neq 0$. Moreover, one can use the Schmidt decomposition of ψ to generate a basis for H^0 .

As an example, if \mathcal{H}_X is the orthonormal \mathbb{C} -span of $|0_X\rangle$ and $|1_X\rangle$ for $X \in \{A, B\}$, and we take ψ to be the Bell-state $\frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$, then

$$H^0(\psi) \cong \text{span}_{\mathbb{C}} \{(|i_A\rangle \langle j_A|, |i_B\rangle \langle j_B|) : i, j = 0, 1\} / \text{span}_{\mathbb{C}} \{(\mathbb{1}_A, \mathbb{1}_B)\};$$

where $\mathbb{1}_X$ denotes the identity operator on \mathcal{H}_X . This can be thought of as a quotient of the vector space of all non-locally correlated pairs of operators (“random variables”) by the one-dimensional subspace spanned by the “trivial” correlated operators (i.e. identity operators).

As an example of what multipartite versions of our complexes can accomplish, we state some results relevant to the tripartite GHZ state and the W-state:

$$\begin{aligned} \text{GHZ} &:= \frac{1}{\sqrt{2}}(|111\rangle + |000\rangle) \in (\text{span}_{\mathbb{C}}\{|0\rangle, |1\rangle\})^{\otimes 3}, \\ \text{W} &:= \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \in (\text{span}_{\mathbb{C}}\{|0\rangle, |1\rangle\})^{\otimes 3}. \end{aligned}$$

Both states are entangled and are known to be in distinct *SLOCC classes*, i.e. equivalence classes under the action of “local automorphisms”: transformations of the form $A_1 \otimes A_2 \otimes A_3 \in \text{GL}_2(\mathbb{C})^{\otimes 3}$ [36]. The tripartite mutual information is invariant under such transformations, and can be thought of as an indicator of information shared among all three tensor factors when non-vanishing. However, it vanishes on both of these states: failing to distinguish their SLOCC equivalence classes, detect entanglement, or detect shared information among all three tensor factors.

Poincaré polynomials associated to cohomologies of tripartite states are also invariant under local automorphisms. Recall that the Poincaré polynomial associated to a cochain complex C is given by:

$$P[C] := \sum_{k \in \mathbb{Z}} y^k \dim_{\mathbb{C}} H^k[C] \in \mathbb{Z}[y].$$

For a state ψ on a tensor product of Hilbert spaces, we can associate two cochain complexes: the *GNS complex* $G(\psi)$ (constructed from the Čech complex associated to a presheaf of GNS representations) and the *commutant complex* $E(\psi)$ (constructed from the Čech complex associated to a presheaf of commutants of GNS representations). The Poincaré polynomials of the GNS complexes of our tripartite states are:

$$P[G(\text{GHZ}_3)] = P[G(W_3)] = 1 + 6y.$$

which do not distinguish SLOCC classes. However, the fact that these polynomials are non-vanishing tells us that there exist tuples of operators that exhibit non-local correlations; something the tripartite mutual information fails to detect. In particular there is a single tuple of 1-body operators exhibiting non-trivial correlations between individual tensor factors, and six linearly independent tuples of 2-body operators that exhibit non-trivial correlations between pairs of tensor factors. Explicit generators for the associated cohomology components can be found in [10].

The resulting Poincaré polynomials for the commutant complexes are:

$$P[E(\text{GHZ})] = 7 + 7y,$$

$$P[E(W)] = 3 + 3y.$$

As a result, our Poincaré polynomials detect the information shared among tensor factors due to entanglement, and distinguish the SLOCC classes of the W and GHZ state.

3. BPS-INDICES/DT-INVARIANTS AND SPECTRAL NETWORKS

In theories with extended supersymmetry, BPS states are states that are invariant under (i.e. annihilated by) half of the supercharges. Such states remain stable (in a suitable sense) as one varies certain parameters or moduli in the theory; for this reason, they have become a powerful tool in probing parts of the spectra of supersymmetric field theories and play a prominent role in understanding the underlying geometry of moduli spaces of supersymmetric vacua [37, 38]. The *BPS-index* [39] (a.k.a. the second helicity supertrace) is a weighted “count” of the number of BPS states that is well-behaved under tuning of vacuum-moduli. Typically in super-symmetric field/string theories, BPS states take on semi-classical descriptions closely related to interesting geometric quantities; in such situations BPS-indices can be thought of as counts of:

- (1) special Lagrangians in some (Calabi-Yau) Calabi-Yau threefold X (“A-branes”);
- (2) coherent sheaves (“B-branes”) on a Calabi-Yau threefold X^\vee (the mirror-dual description to 1);
- (3) moduli of complex quiver representations;

sometimes equivalent (or dual) descriptions of the same theory allow all three interpretations to play a role (e.g. Denef [40] describes a physical derivation of the relationship between special Lagrangians and quivers). The BPS-index, defined as a trace over a finite-dimensional Hilbert space, motivated mathematicians to develop the notion of (generalized) Donaldson-Thomas (DT) invariants [41, 42, 43, 44, 45], defined directly in terms of geometric quantities; in the proper context, DT-invariants and BPS-indices are conjecturally the same.

My thesis is focused on calculating BPS-indices utilizing the *spectral network* technique introduced by Gaiotto-Moore-Neitzke (GMN) [46, 47, 48]. Specifically, the main theme of my thesis pertains to the algebraicity over $\mathbb{Q}(x)$ — hereafter referred to just as *algebraicity* — of (appropriately defined) generating functions for BPS-indices; my main results are:

- (1) explicit algebraic relations obeyed by BPS-index generating functions;
- (2) constraints on the asymptotics of BPS-indices due to algebraicity

Details can be found in [49] and my earlier work with Galakhov, Longhi, Moore, and Neitzke [50].

Algebraicity has implications for cluster varieties – varieties constructed by gluing together algebraic tori via certain birational maps—as the generating functions for DT-invariants (associated to quivers) appear explicitly in such gluing transformations. Cluster varieties are themselves interesting objects: they offer explicit geometric realizations of mirror symmetry and sometimes arise as the total space of an integrable system (e.g. the Hitchin integrable system)—a subject that has a large overlap with supersymmetric field theory.

Further, the asymptotics induced by algebraicity imply that, in some supersymmetric field theories, the number of states of mass $\leq M$ grows exponentially with M – indicating the presence of “maximum” temperatures (*Hagedorn* temperatures) or possible phase transitions for such theories. In my joint work with D. Galakhov, P. Longhi, G. Moore, and A. Neitzke [50], we found the first explicit expressions for the “exponential growth” of BPS-indices (providing a lower-bound on the growth of the density of states) in four-dimensional $\mathcal{N} = 2$, $SU(3)$ Super-Yang Mills. To string-oriented theorists, exponential growth in a field theory (as opposed to a string-theory) was quite unexpected—naïve thermodynamic arguments seem to suggest this growth is impossible [50]—but was a familiar idea to phenomenologists in the context of (standard model) QCD for quite some time [51, 52].

Kontsevich and Soibelman have an understanding of algebraicity for a large class of DT-invariants [53]; however they claim that their proof uses rather indirect methods. Using spectral networks, one can see algebraicity directly, and algorithmically construct explicit algebraic relations obeyed by generating functions. Roughly speaking, a spectral network is a directed, decorated graph associated a \mathbb{Z} -family of BPS states; algebraic relations for generating functions of the associated BPS-indices follows by a system of algebraic relations determined by the edges and vertices of this graph.

3.1. Future Directions. As noted, algebraicity has implications for those studying cluster-varieties. A related consequence is a need to carefully define GMN’s non-abelianization map [48, §10]: a map that (conjecturally) supplies cluster coordinates to a moduli space of $GL_K\mathbb{C}$ -local systems (with flag data) on a fixed punctured (real, smooth) surface and equipped with fixed monodromy around punctures. This moduli space arises as a symplectic leaf of (one of) Fock and Goncharov’s \mathcal{X} -space(s) [54]. I partially sketched a definition of a functorial version of the non-abelianization map where algebraicity plays a central role. The role of Hagedorn temperatures due to exponential BPS-asymptotics is also largely unknown and should be explored further. The complicated nature of spectral networks in theories of class S of rank > 3 (a class of four-dimensional $\mathcal{N} = 2$ field theories), suggest that the exponential BPS-asymptotics in $\mathcal{N} = 2$, $SU(3)$ SYM [50] may be a common feature for a large class of four-dimensional $\mathcal{N} = 2$ field theories.

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