Higher Information from Families of Measures

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Abstract. We define the notion of a measure family: a pre-cosheaf of finite measures over a finite set; every joint measure on a product of finite sets has an associated measure family. To each measure family there is an associated index, or "Euler characteristic", related to the Tsallis deformation of mutual information. This index is further categorified by a (weighted) simplicial complex whose topology retains information about the correlations between various subsystems.

1 Introduction

Questions relating to the independence of random variables have a deep relationship to questions of topology and geometry: given the data of a multipartite, a.k.a. "joint" measure, there is an emergent "space" that encodes the relationship between various subsystems. Topological invariants of this emergent space capture non-trivial correlations between different subsystems: this includes numerical invariants—such as the Euler characteristic—which roughly indicate *how much* information is shared characteristic, as well as "higher" invariants—such as cohomology—that capture *what* information is shared. In [11] these ideas were explored, using a language engineered for an audience interested in the purely quantum regime, i.e., "non-commutative" measure theory. This note provides a sketch of the categorical underpinnings of these ideas in the opposite "classical" or "commutative" extreme, focusing on finite atomic measure spaces for brevity. Some of these underpinnings are partly outlined in the recorded talks [12, 13].

The majority of this note is dedicated to formalizing the working parts that underlie the "commutative diagram" in Figure 1. The word "space" is taken to mean a (semi-)simplicial measure or a (weighted semi-)simplicial set, and the grayed out mystery box indicates a suspected "weighted" version of cohomology that may provide a novel measure of shared information. The classical picture that is presented here is unified with the quantum picture of [11] using the language of von Neumann algebras.[1] A reader wishing to learn how this fits into a larger picture should consult [11] and the talks [12, 13]. The upcoming paper [9] is a related spin-off of the categorical and W*-algebraic underpinnings of some ideas discussed here.

Our categorical perspective of measures on finite sets has close ties to the work of Baez, Fritz, and Leinster [2, 3], and the quantum mechanical generalization is

[1] See [15] for a precise categorical equivalence between commutative W*-algebras and (localizable) measurable spaces.

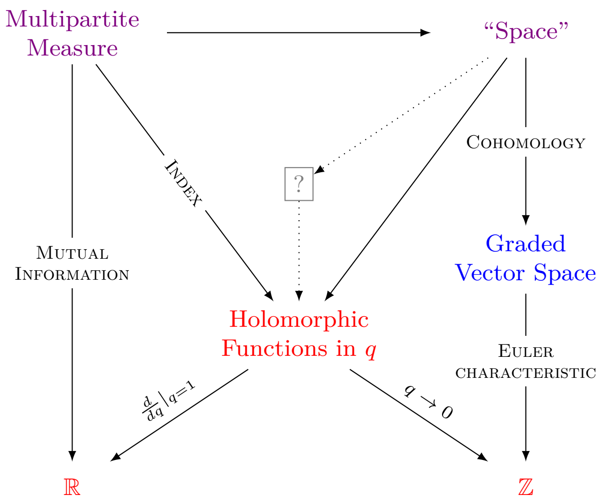

Fig. 1. A "commutative diagram" summarizing the big picture behind the definitions and results stated in this note.

related to the work of Parzygnat [14]. The homotopical or homological perspective has strong relations to the work of Baudot and Bennequin [4]; Vigneaux [18]; Sergeant-Perthuis [16]; and Drummond-Cole, Park, and Terilla [7,8]. Ideas around the index (§4.3) bear relation to the work of Lang, Baudot, Quax, and Forré [10].

2 Preliminaries

In this note, a (finite) *measure* μ consists of the data of a finite set Ω_μ and a function $\mathrm{Subset}(\Omega_\mu) \to \mathbb{R}_{\geq 0}$ that evaluates to zero on $\emptyset$ and satisfies the *additivity condition*: the value on a subset U reduces to the sum of its evaluation on points of U. In a mild abuse of notation, we use μ to denote the function $\mathrm{Subset}(\Omega_\mu) \to \mathbb{R}_{\geq 0}$. We allow for measures to be identically zero on a set, and also allow for the *empty measure*: the unique measure on the empty set.[2]

If μ and ν are measures with $\Omega_\mu = \Omega_\nu = \Omega$ we write $\mu \leq \nu$ if $\mu(U) \leq \nu(U)$ for all $U \subseteq \Omega$. Given a measure μ, and a function between sets $\underline{f}\colon \Omega_\mu \to \Gamma$, the *pushforward measure* $\underline{f}_*\mu$ is the measure with set $\Omega_{\underline{f}_*\mu} := \Gamma$ and $(\underline{f}_*\mu)(U) := \mu[\underline{f}^{-1}(U)]$ for any $U \subseteq \Gamma$. When $\underline{f}_*\mu = \nu$ we call $\underline{f}$ *measure-preserving*.

3 The Category of Finite Measures

Definition 1. Meas *is the category with objects given by finite measures, and a morphism* $f\colon \mu \to \nu$ *defined by an underlying function on sets* $\underline{f}\colon \Omega_\mu \to \Omega_\nu$ *such that* $\underline{f}_*\mu \leq \nu$.

[2] The empty measure corresponds to the zero expectation value on the zero algebra.

Remark 1. Meas utilizes the relation $\leq$ to define a larger class of morphisms than the similarly named category in the work of Baez-Fritz-Leinster [2,3], who define morphisms as measure-preserving functions. Nevertheless, isomorphisms are measure-preserving bijections (Lemma 2); so the notion of isomorphism coincides with that of Baez-Fritz-Leinster.

Lemma 1. Meas *has:*

1. *A symmetric monoidal structure induced by the product of underlying sets: Let μ and ν be measures, then $\mu \otimes \nu$ is the product measure on $\Omega_\mu \times \Omega_\nu$.*
2. *Coproduct $\boxplus$ induced by the disjoint union $\coprod$ of sets: for any measures μ and ν, $\Omega_{\mu \boxplus \nu} := \Omega_\mu \coprod \Omega_\nu$ and $(\mu \boxplus \nu)(U) := \mu(U \cap \Omega_\mu) + \nu(U \cap \Omega_\nu)$ for any $U \subseteq \Omega_\mu \coprod \Omega_\nu$.*

Proof. Verifying that $\otimes$ provides a symmetric monoidal structure is straightforward. To see that $\boxplus$ is a coproduct, note that the inclusion map $\underline{\iota}_\mu \colon \Omega_\mu \to \Omega_\mu \coprod \Omega_\nu$ defines a valid morphism $\underline{\iota}_\mu \colon \mu \to \mu \boxplus \nu$ as $(\underline{\iota}_\mu)_* \mu = \mu \boxplus 0_{\Omega_\nu} \leq \mu \boxplus \nu$. Similarly, $\underline{\iota}_\nu \colon \Omega_\nu \to \Omega_\mu \coprod \Omega_\nu$ defines a morphism $\iota_\nu \colon \nu \to \mu \boxplus \nu$. The universal property follows in part by the fact that $\coprod$ is a coproduct for sets.

Remark 2. The operation $\otimes$ is a categorical product in Meas, but it is *not* a categorical product in the quantum-classical enlargement of Meas.

Remark 3. The fact that $\boxplus$ is a coproduct relies on the presence of non-probability measures and maps that are not measure preserving.[3]

3.1 The Rig of Isomorphism Classes of Measures

The following lemma is straightforward.

Lemma 2. *$f \colon \mu \to \nu$ is an isomorphism if and only if $\underline{f}$ is a bijection and $\underline{f}_* \mu = \nu$.*

Remark 4. One can generalize the class of morphisms in Meas to the class of stochastic maps that manifest algebraically as completely positive contractions on $*$-algebras of $\mathbb{C}$-valued random variables. Even in this situation, isomorphisms would still be measure-preserving bijections.

The collection of isomorphism classes [Meas] of Meas is a set.[4] Moreover, letting $[\mu]$ denote the isomorphism class of an object μ, we can define binary operations $+$ and $\cdot$ by: $[\mu] + [\nu] := [\mu \boxplus \nu]$ and $[\mu] \cdot [\nu] := [\mu \otimes \nu]$ for any pair of objects μ and ν. These equip [Meas] with the structure of a commutative *rig*: a

[3] If one works with probability measures and measure-preserving maps, $\boxplus$ instead manifests as an operadic structure which encapsulates the ability to take convex linear combinations of probability measures; this is the approach taken by [6].

[4] It is easy to write down a natural bijection of [Meas] with $\coprod_{n=0}^{\infty} (\mathbb{R}_{\geq 0})^{\times n}$, taking $\mathbb{R}^{\times 0} := \{\star\}$ to correspond to the empty measure. This observation can be used to equip [Meas] with a topology as in [3].

commutative ring dropping the condition that there are additive inverses (a ring without "negatives"). The empty measure $\emptyset$ provides an additive (+) unit, and the unit measure on the one-point set provides a multiplicative ($\cdot$) unit.

The following theorem is a take on the observations of Baez-Fritz-Leinster in [2].

Theorem 1. *Let $\mathcal{O}(\mathbb{C})$ denote the ring of holomorphic functions on $\mathbb{C}$. There is a unital homomorphism* $\dim\colon [\mathsf{Meas}] \to \mathcal{O}(\mathbb{C})$*, defined on any* $[\mu]$ *by:*

$$\begin{aligned} \dim[\mu]\colon \mathbb{C} &\longrightarrow \mathbb{C} \\ q &\longmapsto \sum_{\omega \in \Omega_\mu} \mu(\{\omega\})^q =: \dim_q[\mu], \end{aligned}$$

where, for any $\lambda \geq 0$, $\lambda^0 := \lim_{q\to 0} \lambda^q$: i.e., $\lambda^0 = 1$ if $\lambda > 0$ and $0^0 := 0$.

Remark 5. There is a reflective full subcategory of Meas generated by "faithful" measures: measures μ such that $\mu(\{\omega\}) > 0$ for all $\omega \in \Omega_\mu$. The homomorphism dim is an isomorphism on this full subcategory (see [2]).

Remark 6. An extension of Theorem 1 to finite-dimensional quantum-classical systems appears in the constructions of [11, §8.4.1].

Remark 7. The parameter q in $\dim_q$ has several potential interpretations:

1. As a character of a continuous complex irreducible representation of the multiplicative group $\mathbb{R}_{>0}$: every such representation is of the form $m_q\colon \mathbb{R}_{>0} \to \mathrm{Aut}(\mathbb{C})$ for some $q \in \mathbb{C}$ such that $m_q(\lambda)z = \lambda^q z$.
2. As a (negative) inverse temperature: $\dim_{-\beta}[\mu]$ is the partition function $\sum_{\omega\in\Omega_\mu} e^{-\beta E(\omega)}$, associated to the classical system with state space Ω_μ and energy function $E\colon \Omega_\mu \to \mathbb{R}$ given by $E(\omega) := \log[\mu(\{\omega\})]$.
3. As the parameter defining a q-norm for an L^q space.

A detailed justification for the first and third interpretation is left for future work. The second interpretation is is also discussed in [2].

4 Measure Families

Definition 2. *A measure family* $\mathsf{\mu}$ *is the data of a finite set* $P_{\mathsf{\mu}}$ *and a functor*

$$\mathsf{Subset}(P_{\mathsf{\mu}}) \longrightarrow \mathsf{Meas},$$

where $\mathsf{Subset}(P_{\mathsf{\mu}})$ *is the category with objects given by subsets of* $P_{\mathsf{\mu}}$*, and a unique morphism $T \to V$ if and only if $T \subseteq V$. Analogous to the situation for measures, we abuse notation and denote the functor by* $\mathsf{\mu}$*.*

Given a function between finite sets $\underline{f}\colon P_{\mathsf{\mu}} \to Q$, we can define the pushforward of a measure family $\mathsf{\mu}$ as the measure family $\underline{f}_*\,\mathsf{\mu}\colon \mathsf{Subset}(Q) \to \mathsf{Meas}$ defined by

$$(\underline{f}_*\,\mathsf{\mu})(T) := \mathsf{\mu}[\underline{f}^{-1}(T)]$$

for every $T \subseteq Q$. Given two measure families $\mathsf{\mu}$ and $\mathsf{\nu}$ with $P = P_{\mathsf{\mu}} = P_{\mathsf{\nu}}$, we say $\mathsf{\mu} \leq \mathsf{\nu}$ if and only if $\Omega_{\mathsf{\mu}(T)} = \Omega_{\mathsf{\nu}(T)}$ and $\mathsf{\mu}(T) \leq \mathsf{\nu}(T)$ for every $T \subseteq P$.

Definition 3. *The category of measure families* $\mathsf{MeasFam}$ *is the category with objects given by measure families, and a morphism* $f\colon \mu \to \nu$ *defined by a function* $\underline{f}\colon P_\mu \to P_\nu$ *such that* $\underline{f}_* \mu \le \nu$.

There are various versions of "lifts" of the monoidal operations $\boxplus$ and $\otimes$ on Meas to monoidal operations on $\mathsf{MeasFam}$; the following versions will be useful.

Definition 4. *Let* μ *and* ν *be measure families, then* $\mu \boxplus \nu$ *and* $\mu \otimes \nu$ *are measure families with* $P_{\mu\boxplus\nu}$ *and* $P_{\mu\otimes\nu}$ *both defined as the disjoint union* $P_\mu \coprod P_\nu$*. On a subset* $T \subseteq P_\mu \coprod P_\nu$ *we define* $(\mu \boxplus \nu)(T) \coloneqq \mu(T \cap P_\mu) \boxplus \nu(T \cap P_\nu)$*, and* $(\mu \otimes \nu)(T) \coloneqq \mu(T \cap P_\mu) \otimes \nu(T \cap P_\nu)$*. The definitions on inclusions follow from the obvious induced morphisms.*

Definition 5. *Let* μ *be a measure family, and* $T \subseteq P_\mu$*; then* $\mu|_T\colon \mathsf{Subset}(T) \to \mathsf{Meas}$ *denotes the obvious restriction. We say* μ *is a 2-measure if* $\mu(\emptyset)$ *is the empty measure and there is an isomorphism* $\mu \xrightarrow{\sim} \boxplus_{p\in P} \mu|_{\{p\}}$.

2-measures are measure families where all global data is given by gluing together local data.[5] This is a categorified notion of the additivity condition for a measure.

4.1 2-Measures from Measures

Let μ be a measure, then there is a measure family $\mathtt{R}^\mu\colon \mathsf{Subset}(\Omega_\mu) \to \mathsf{Meas}$ given by the restriction of μ to subsets of P. On objects, it acts in the following way: for $T \subseteq \Omega_\mu$ nonempty, $\mathtt{R}^\mu\, T \coloneqq \mu|_T$, where $\mu|_T$ the restriction of μ to subsets of T; to the empty set we assign the empty measure. To every inclusion $T \subseteq V$, it assigns the morphism $\mathtt{R}^\mu\, T \to \mathtt{R}^\mu\, V$ whose underlying map is the inclusion map $T \hookrightarrow V$. The additivity condition on a measure requires that for any subset $T \subseteq P$ the identity map $T \to T$ induces an isomorphism of measures: $\mathtt{R}^\mu\, T \xrightarrow{\sim} \boxplus_{t\in T} \mathtt{R}^\mu(\{t\})$. As a result, $\mathtt{R}^\mu$ is a 2-measure. Conversely, any 2-measure that reduces to a coproduct of measure families on one point sets defines a measure on P_μ.

4.2 Measure Families From Multipartite Measures

Definition 6. *A multipartite measure* $\boldsymbol{\mu}$ *over a finite subset* P *is a collection of sets* $\{\Omega_p\}_{p\in P}$*, and a measure* $\mu_P\colon \mathrm{Subset}(\prod_{p\in P} \Omega_p) \to \mathbb{R}_{\ge 0}$.

Given a multipartite measure $\boldsymbol{\mu}$ over P, define Ω_T as $\prod_{t\in T} \Omega_t$ if $T \ne \emptyset$, and the one-point set $\{\star\}$ if $T = \emptyset$. Let $\underline{p}_T\colon \Omega_P \to \Omega_T$ denote the projection map if $T \ne \emptyset$ and the map to the point otherwise. Then to each subset $T \subseteq P$, we can assign a *reduced* (or "marginal") measure $\mu_T \coloneqq (\underline{p}_T)_* \mu_P$.

The data of the reduced measures collects into a functor $\mathsf{Subset}(P)^{\mathrm{op}} \to \mathsf{Meas}$ that takes a subset T of P to μ_T, and takes an inclusion $T \subseteq V$ to the morphism $\mu_V \to \mu_T$ provided by the projection $\Omega_V \to \Omega_T$. Because this functor is contravariant, it is *not* a measure family; however, we can make it one by

[5] In some sense a 2-measure is an "acyclic cosheaf" of measures.

composing with the functor $(-)^c_P\colon \mathsf{Subset}(\Omega)^{\mathrm{op}} \to \mathsf{Subset}(\Omega)$ that takes a subset of P to its complement. The result is a measure family

$$\mathtt{A}^{\boldsymbol{\mu}}\colon \mathsf{Subset}(P) \longrightarrow \mathsf{Meas},$$

which acts on objects by taking T to μ_{T^c}. Factorizability questions about $\boldsymbol{\mu}$ are equivalent to factorizability questions about $\mathtt{A}^{\boldsymbol{\mu}}$: e.g. $\mu_P = \bigotimes_{p\in P} \mu_p$, if and only if there is an isomorphism $\mathtt{A}^{\boldsymbol{\mu}} \xrightarrow{\sim} \bigotimes_{p\in P} \mathtt{A}^{\boldsymbol{\mu}}|_{\{p\}}$.

4.3 The Index

Definition 7. *Let* $\upmu$ *be a measure family. The index of* $\upmu$ *is defined as the holomorphic function*

$$\mathfrak{X}[\upmu] \coloneqq \sum_{k=0}^{|P_\upmu|} (-1)^k \dim \left[\boxplus_{|T|=k} \upmu(T^c) \right].$$

The evaluation of $\mathfrak{X}[\upmu]$ *at* $q \in \mathbb{C}$ *is denoted as* $\mathfrak{X}_q[\upmu]$.

The complement in the definition is for convenience;[6] without it, the definition would be the same up to the overall sign $(-1)^{|P_\mu|}$. Theorem 1 and manipulations of the inclusion-exclusion relation defining the index lead to the following results.

Theorem 2. *The index only depends on isomorphism classes of measure families; moreover,* $\mathfrak{X}[\upmu \otimes \upnu] = \mathfrak{X}[\upmu]\mathfrak{X}[\upnu]$ *for any measure families* $\upmu$ *and* $\upnu$.

Proposition 1. *If* $\upmu$ *and* $\upnu$ *measure families with* $P_\upmu$ *and* $P_\upnu$ *non-empty, then* $\mathfrak{X}_q[\upmu \boxplus \upnu] = 0$. *In particular,* $\mathfrak{X}$ *vanishes on any 2-measure* $\upmu$ *with* $|P_\upmu| \geq 2$.

According to Proposition 1, a non-vanishing index indicates that there is an obstruction to an "additive" ($\boxplus$) descent of data. For a multipartite measure $\boldsymbol{\mu}$, we are more interested in an obstruction to a "*multiplicative*" ($\otimes$) descent of data, i.e., a failure to factorize. As the discussion below indicates, this can be detected by looking at the derivative of $q \mapsto \mathfrak{X}_q[\mathtt{A}^{\boldsymbol{\mu}}]$ at $q = 1$ (where $\mathfrak{X}_1[\mathtt{A}^{\boldsymbol{\mu}}] = 0$).

Tsallis Mutual Information and the $q \to 1$ Limit For a multipartite measure $\boldsymbol{\mu}$ we have:

$$\mathfrak{X}_q[\mathtt{A}^{\boldsymbol{\mu}}] = \sum_{\emptyset\subseteq T\subseteq P} (-1)^{|T|} \left(\sum_{\omega\in\Omega_T} \mu_T(\{\omega\})^q \right).$$

If $\boldsymbol{\mu}$ is a multipartite probability measure ($\mu_P(P) = 1$), then a bit of manipulation demonstrates that $I_q[\boldsymbol{\mu}] \coloneqq \frac{1}{q-1}\mathfrak{X}_q[\mathtt{A}^{\boldsymbol{\mu}}]$ can be rewritten as:

$$I_q[\boldsymbol{\mu}] = \sum_{\emptyset\neq T\subseteq P} (-1)^{|T|-1} S_q^{\mathrm{Ts}}(\mu_T),$$

[6] One can define an index with respect to any cover of P_μ; but our primary interest will be the cover that is the complement of the finest partition of $P_\upmu$.

where $S_q^{\mathrm{Ts}}(\mu) := \frac{1}{q-1}[1 - \sum_{\omega \in \Omega} \mu(\{\omega\})^q]$ is the Tsallis deformation of mutual information. Multipartite mutual information is recovered in the limit that $q \to 1$. If $\boldsymbol{\mu}$ is a multipartite measure on a one-element set, then $I_q[\boldsymbol{\mu}]$ is simply the Tsallis entropy. This observation can be combined with multiplicativity of the index (Theorem 2), to demonstrate that the multipartite mutual information of a multipartite measure on P must vanish if the measure factorizes with respect to any partition of P: see [11, §8.5].

The $q \to 0$ Limit $\dim_0 \mu$, which is $\lim_{q \to 0} \dim_q \mu$ by definition, is an integer counting the number of points of Ω_μ with non-vanishing measure. Thus, for $\upmu$ any measure family, $\mathfrak{X}_0[\upmu]$ is an integer. This is a hint that $\mathfrak{X}_0[\upmu]$ is related to the Euler characteristic of a topological space.

4.4 (Semi-)Simplicial Objects

By viewing a measure family $\upmu$ as a pre-cosheaf and applying Čech techniques with respect to a cover of $P_\upmu$, we can construct an (augmented)[7] semi-simplicial object in Meas: an (augmented) *semi-simplicial measure*. In this note, we specialize to the "complementary cover" $\{\{p\}^c\}_{p \in P_\upmu}$ of $P_\upmu$ and choose a total order on P.[8] Using the fact that the intersection of complements is the complement of a union, the non-trivial part of the resulting augmented semi-simplicial measure can be summarized by a diagram in Meas of the form:

$$\overbrace{\upmu(\emptyset^c)}^{\text{Degree } -1} \longleftarrow \overbrace{\boxplus_{|T|=1} \upmu(T^c)}^{\text{Degree } 0} \leftleftarrows \cdots \underbrace{\vdots}_{n-1 \text{ arrows}} \overbrace{\boxplus_{|T|=n-1} \upmu(T^c)}^{\text{Degree } n-2} \underbrace{\vdots}_{n \text{ arrows}} \overbrace{\upmu(P_\upmu^c)}^{\text{Degree } n-1} , \quad (1)$$

where: $n = |P|$, degree $\geq n$ components are taken to be the empty measure, and the arrows satisfy the face-map relations of an augmented semi-simplicial object. The holomorphic function $-\mathfrak{X}[\upmu]$, where $\mathfrak{X}[\upmu]$ is the index of Definition 7, can be thought of as the graded dimension or "Euler characteristic" of (1).

Remark 8. In the special case that $\upmu = \mathtt{A}^{\boldsymbol{\mu}}$ for a multipartite measure $\boldsymbol{\mu}$ over P, the complements disappear: for any $T \subseteq P$, we have $\mathtt{A}^{\boldsymbol{\mu}}(T^c) = \mu_T$ (the reduced measure on $\prod_{t \in T} \Omega_t$). With this specialization, diagram (1) becomes:

$$\overbrace{\mu_\emptyset}^{\text{Degree } -1} \longleftarrow \overbrace{\boxplus_{|T|=1} \mu_T}^{\text{Degree } 0} \leftleftarrows \cdots \vdots \overbrace{\boxplus_{|T|=|P|-1} \mu_T}^{\text{Degree } |P|-2} \vdots \overbrace{\mu_P}^{\text{Degree } |P|-1} , \quad (2)$$

where, as before, we ignore the empty measures in degree $\geq |P|$ components. The augmentation map into the degree -1 component has underlying map given by

[7] Augmented in this context means there is an additional degree -1 component and a single map from the degree 0 component to the degree -1 component.

[8] All interesting quantities are equivariant under change of total order.

the unique map $\coprod_{p \in P} \Omega_p \to \{\star\} = \Omega_\emptyset$, and the remaining face maps are induced by projection maps composed with inclusions into disjoint unions. For instance, if $P = \{1, 2\}$, the two face maps out of the degree 1 component have underlying maps given by the following compositions (letting $i \in \{1, 2\}$):

$$\Omega_1 \times \Omega_2 \xrightarrow{\text{project}} \Omega_i \xhookrightarrow{\text{include}} \Omega_1 \coprod \Omega_2.$$

Remark 9. One can also apply Čech techniques to produce (augmented) simplicial objects rather than (augmented) *semi*-simplicial objects (see, [5] and [1, §25.1 - 25.5]). Simplicial objects include additional "degeneracies" and extend the diagram (1) infinitely far to the right with possibly non-empty measures. The invariants of the underlying measure family that are discussed in this note—Euler characteristics, indices, and cohomology—can be recovered by passing through either version: yielding results that are equivalent, or canonically isomorphic. This note focuses on the semi-simplicial version for pedagogical reasons and immediate connections to the computational underpinnings of [11].

(Semi-)Simplicial Sets From the semi-simplicial measure (1), one can derive an (augmented) semi-simplicial set: a slight generalization of a simplicial complex. Indeed, there is a functor:

$$\mathtt{S} \colon \mathsf{FinMeas} \longrightarrow \mathsf{FinSet}$$

that assigns to a measure μ, its support:

$$\mathtt{S}\,\mu \coloneqq \{\omega \in \Omega_\mu \colon \mu(\omega) \neq 0\},$$

and assigns to a morphism $f \colon \mu \to \nu$, the morphism $\mathtt{S}\,f \colon \mathtt{S}\,\mu \to \mathtt{S}\,\nu$ whose underlying function on sets is the restriction $\underline{f}|_{\mathtt{S}\,\mu}$: a valid assignment as $\underline{f}(\mathtt{S}\,\mu) \subseteq \mathtt{S}\,\nu$ due to the condition $\underline{f}_*\mu \leq \nu$. Applying $\mathtt{S}$ to our semi-simplicial measure, we obtain an augmented semi-simplicial set $\Delta[\mathsf{\mu}]$ whose non-trivial part is summarized by the following diagram in FinSet (with $n = |P_{\mathsf{\mu}}|$):

$$\underbrace{\mathtt{S}\,\mathsf{\mu}(\emptyset^c)}_{\text{Degree } -1} \longleftarrow \underbrace{\coprod_{|T|=1} \mathtt{S}\,\mathsf{\mu}(T^c)}_{\text{Degree } 0} \leftleftarrows \cdots \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{smallmatrix} \underbrace{\coprod_{|T|=n-1} \mathtt{S}\,\mathsf{\mu}(T^c)}_{\text{Degree } n-2} \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{smallmatrix} \underbrace{\mathtt{S}\,\mathsf{\mu}(P_{\mathsf{\mu}}^c)}_{\text{Degree } n-1}.$$

The Euler characteristic of $\Delta[\mathsf{\mu}]$, denoted $\chi(\Delta[\mathsf{\mu}])$, is the negative of the $q = 0$ evaluation (or $q \to 0$ limit) of the index:

$$\chi(\Delta[\mathsf{\mu}]) \coloneqq \sum_{k=-1}^{|P_{\mathsf{\mu}}|-1} (-1)^k \left(\sum_{|T|=k+1} |\,\mathtt{S}\,\mathsf{\mu}(T^c)| \right) = -\mathfrak{X}_0[\mathsf{\mu}].$$

Geometric Realizations Specialize to $\mathsf{\mu} = \mathtt{A}^{\boldsymbol{\mu}}$, and define $\Delta_{\boldsymbol{\mu}} \coloneqq \Delta[\mathtt{A}^{\boldsymbol{\mu}}]$, which is summarized by the diagram:

$$\mathtt{S}\,\mu_\emptyset \longleftarrow \coprod_{|T|=1} \mathtt{S}\,\mu_T \leftleftarrows \cdots \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{smallmatrix} \coprod_{|T|=|P|-1} \mathtt{S}\,\mu_T \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{smallmatrix} \mathtt{S}\,\mu_P.$$

Let $\Delta'_{\boldsymbol{\mu}}$ denote the semi-simplicial set obtained by removing the augmentation map[9] into $\mathtt{S}\,\mu_\emptyset$. To construct the geometric realization $|\Delta'_{\boldsymbol{\mu}}|$, observe that 0-simplices are given by points $\omega \in \coprod_{p\in P} \Omega_p$ such that $\mu_{\{p\}}(\{\omega\}) \neq 0$ for every $p \in P$. Higher k-simplices are given by collections of 0-simplices with non-vanishing measure as computed with respect to the reduced measure $\boxplus_{|T|=k+1} \mu_T$. The geometric realization is simple: first identify $P = P_{\mathtt{A}^{\boldsymbol{\mu}}}$ with the set $\{1, \cdots, n\}$, then for each $(\omega_1, \cdots, \omega_n) \in \prod_{i=1}^n \Omega_i$ with $\mu(\{(\omega_1, \cdots, \omega_n)\}) \neq 0$, draw an $(n-1)$-simplex with vertices $\omega_1, \omega_2, \ldots, \omega_n$.

When $\boldsymbol{\mu}$ is a bipartite measure on the set $P = \{1, 2\}$, the geometric realization $|\Delta'_{\boldsymbol{\mu}}|$ might look familiar: it is a bipartite (directed) graph whose vertices are colored by points in P. A bit of experimentation demonstrates that the connectivity of this graph is closely related to the correlations between "subsystems" 1 and 2.

Cohomology If we apply the functor[10] $\mathrm{Hom}_{\mathsf{FinSet}}(-, \mathbb{C})\colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{FinVect}_{\mathbb{C}}$ to $\Delta_{\mathsf{\mu}}$, we obtain a cosimplicial vector space; this can can be turned into a cochain complex by taking alternating sums of face maps. For $\mathsf{\mu} = \mathtt{A}^{\boldsymbol{\mu}}$, this complex looks like (letting $\mathbb{C}[-]$ be shorthand for $\mathrm{Hom}_{\mathsf{FinSet}}(-, \mathbb{C})$):

$$\overbrace{\mathbb{C}}^{\text{Degree -1}} \longrightarrow \overbrace{\prod_{|T|=1} \mathbb{C}[\mathtt{S}\,\mu_T]}^{\text{Degree 0}} \longrightarrow \overbrace{\prod_{|T|=2} \mathbb{C}[\mathtt{S}\,\mu_T]}^{\text{Degree 1}} \longrightarrow \cdots \longrightarrow \overbrace{\mathbb{C}[\mathtt{S}\,\mu_P]}^{\text{Degree } |P|-1} \longrightarrow 0 \longrightarrow \cdots$$

The cohomology $H^\bullet$ of this cochain complex is the *reduced* simplicial cohomology of the geometric realization $|\Delta'_{\boldsymbol{\mu}}|$ with coefficients in $\mathbb{C}$. Representatives of H^k can be interpreted as assignments of $\mathbb{C}$-valued random variables to all subsystems (subsets of $P_{\mathtt{A}^{\boldsymbol{\mu}}}$) of size $k+1$, such that these assignments have linear correlations that do not reduce to correlations on subsystems of size k. Random variables on the subsystem of size 0, coming from the degree -1 component, are the constant random variables. For a bipartite measure on $P = \{1, 2\}$, a representative of a non-zero element in H^0 is a pair (r_1, r_2) of random variables such that (for $i \in \{1, 2\}$): r_i is a random variable on Ω_i that is not almost everywhere (a.e.) equal to zero, r_i is non-constant, and $r_1 \otimes 1 - 1 \otimes r_2$ is a.e. equal to zero with respect to the measure on $\Omega_1 \times \Omega_2$. See [11, §7.4] and [11, §6.5] for interpretations of the quantum mechanical analogs of H^0 and higher H^k; these interpretations can be translated into precise statements for the classical context of this note.[11]

Remark 10. As a source of new invariants of multipartite measures, one might also study the cohomology *ring* of $|\Delta_{\boldsymbol{\mu}}|$ with coefficients in a commutative ring R: a graded R-algebra. In fact, the story in this section can be souped-up to use part of the monoidal Dold-Kan correspondence in order to produce a differential graded R-algebra from a measure family.

[9] If μ_P does not vanish everywhere, then $|\,\mathtt{S}\,\mu_\emptyset| = |\{\star\}| = 1$. Consequently, one can show that $\mathfrak{X}_0[\mathtt{A}^{\boldsymbol{\mu}}] = 1 - \chi(\Delta'_{\mathsf{\mu}})$.

[10] This is a specialization of a functor from (localizable) measurable spaces to the Banach space underlying the W*-algebra of essentially bounded measurable functions.

[11] The classical analog of the GNS and the commutant complexes of [11] both reduce to the alternating sum of face maps complex that is used in this note.

5 Some Future Directions

The reduced measures associated to a multipartite measure supply a "weight" to the simplices of its associated semi-simplicial set. Thus, in some sense, the index is a weighted Euler characteristic. It is natural to suspect that there is a weighted version of cohomology categorifying the index for all values of q: the mystery box of Figure 1. Moreover, the existence of a canonically associated semi-simplicial set to a multipartite measure may open the door to measures of shared information using combinatorial invariants, such as the Stanley-Reisner ring (the "face ring of a simplicial complex" in [17]).

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