A Probability Talk That Spaces Out

ASU
Diff. Geo.
Seminar
1/18/2018

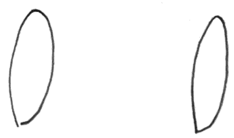
I. Vague Fuzzy Inspiration

Quantum States of d-dim.ˡ Theories ⟷ (d+1)-dim.ˡ Spacetime Geometry

(d: Space dim.) (Theories: CFTs)

Ex:

$\psi = \psi_A \otimes \psi_B \in \mathcal{H}_A \otimes \mathcal{H}_B$ ⟷

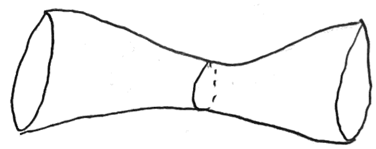


A B

A B

"Disconnected" Space

$\psi = \frac{1}{\sqrt{2}}\left(x_A \otimes x_B + y_A \otimes y_B\right)$ ⟷



"Wormhole"

"Entropies / Mutual Information" ⟷ Area of Surface of Black Hole

II. Something More Precise

Entanglement / GNS Cohomology → Quantum States on $\bigotimes_{f \in F} \mathcal{H}_i$ ⟷ Non-Commutative Geometry (non-commutative algebra)

This Talk → Probability Measures on $\prod_{f \in F} \Omega_f$ ⟷ Geometry (commutative algebra)

Have on board ahead of time

Why Probability Measures?

	Quantum	Classical
Random Variables	$\mathrm{End}(\mathcal{H})$	$\mathrm{Fun}(\Omega, \mathbb{C})$ Ω a Measurable Space
	$\mathcal{H}$ Finite dim$^{\underline{l}}$	Ω a Finite Set
Expectation Values	$\mathbb{E} : \mathrm{End}(\mathcal{H}) \longrightarrow \mathbb{C}$ Positive linear Functional $\mathbb{E} = \mathrm{Tr}[\hat{\rho}\,(-)]$ ($\hat{\rho}$: Positive operator)	$\mathbb{E} : \mathrm{Fun}(\Omega, \mathbb{C}) \longrightarrow \mathbb{C}$ Positive linear Functional $\mathbb{E} = \int_\Omega (\;)\, d\mu$ or $\sum_{\omega \in \Omega} (\,)\mu_\omega$, $\mu : \Omega \to \mathbb{R}_{\geq 0}$
"Pure States"	$\hat{\rho} = \psi \otimes \psi^\vee$	$\mu = \delta_\omega$
"Factorizability"	$\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$ $(\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B)$	$\mu_{AB} = \mu_A \times \mu_B : \Omega_A \times \Omega_B \to \mathbb{R}_{\geq 0}$ $: (a, b) \mapsto \mu_A(a)\mu_B(b)$
	$\mathrm{Tr}[\hat{\rho}] = 1$	$\sum_{\omega \in \Omega} \mu_\omega = 1$

Def^{n}: Reduced Measure / Marginal Measure

Let $\mu_{AB} : \Omega_A \times \Omega_B \longrightarrow \mathbb{R}_{\geq 0}$ be a (Prob) Measure, then we can define:

$$\mu_A : \Omega_A \longrightarrow \mathbb{R}_{\geq 0}$$
$$a \longmapsto \sum_{b \in \Omega_A} \mu_A(a, b)$$

(More generally if $\mu : \prod_{f \in F} \Omega_f \longrightarrow \mathbb{R}_{\geq 0}$ then for any $T \subseteq F$

$$\mu_T : \underset{\Omega_T := \prod_{t \in T} \Omega_t}{r} \longmapsto \sum_{\ell \in pr_T^{-1}(r)} \mu(\ell) \quad ; \; pr_T : \Omega_F \to \Omega_T \;)$$

Claim: Let $\mu : \Omega_1 \times \cdots \times \Omega_n \longrightarrow \mathbb{R}_{\geq 0}$ ($= \mu_{1 \cdots n}$) be a measure on a product of finite sets, then there is a topological space whose "topology" (e.g. fundamental groups, cohomology, ...) detects the failure of $\mu = \mu_1 \times \cdots \times \mu_n$.

Why?

Factorizability of $\mu_{AB} : \Omega_A \times \Omega_B \longrightarrow \mathbb{R}_{\geq 0}$ $\longleftrightarrow$ $\underbrace{\mu_{AB}(a,b) - \mu_A(a)\mu_B(b)}_{C(a,b)} = 0 \quad \forall (a,b)$

$\Updownarrow$

Failure of Factorizability $\longleftrightarrow$ $\exists (a,b)$ w/ $C(a,b) \neq 0$.

But $C \equiv 0$ $\Leftrightarrow$ all "global information" can be obtained by gluing together local data

$$\mathbb{E}_{AB}\left(\sum_{i,j} f_i^A \otimes f_j^B\right) = \sum_{i,j} \mathbb{E}_A(f_i^A)\, \mathbb{E}_B(f_j^B)$$

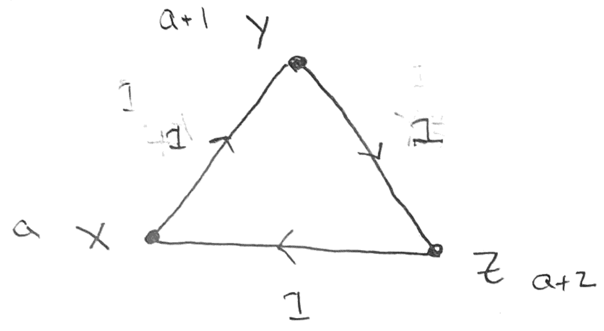
$\Downarrow$

(a,b) w/ $C(a,b) \neq 0$ "obstruct" global descent

Example of Obstruction:

- There is no $\theta : S^1 \longrightarrow \mathbb{R}$ despite the fact that there is a 1-Form $d\theta$.

- Simplicial Version:



$$\mathrm{Span}_{\mathbb{R}}\{X, Y, Z\} \leftarrow \text{Functions on Vertices}$$

$$\big\downarrow d$$

$$\mathrm{Span}_{\mathbb{R}}\{\vec{XY}, \vec{YZ}, \vec{XZ}\} \leftarrow \text{Functions on edges}$$

$\vartheta(\vec{AB}) = 1$ is not in the image of d.

$$dg(\vec{AB}) = g(B) - g(A)$$

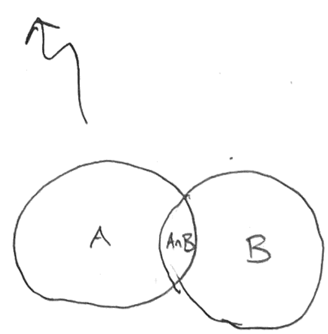
II. Mutual Information as an Euler Characteristic

<u>Def</u>: Let $\mu_{AB} : \Omega_A \times \Omega_B \longrightarrow \mathbb{R}_{\geq 0}$, then the mutual info. of μ_{AB} is

$$I(\mu_{AB}, \{A, B\}) = S(\mu_A) + S(\mu_B) - S(\mu_{AB})$$

Where

$$S(\mu) = \sum_{\omega \in \Omega} \mu_\omega \log \mu_\omega$$



<u>Thm.</u> $I \geq 0$ w/ $I = 0 \iff \mu_{AB} = \mu_A \times \mu_B$

Multivariate Mutual Info.

Let $\mu : \prod_{f \in F} \Omega_f \longrightarrow \mathbb{R}_{\geq 0}$, then

$$I(\mu; F) = - \sum_{T \subseteq F} (-1)^{|F|-|T|} S(\mu_T)$$

$$= (-1)^{|F|+1} \sum_{k=0}^{|F|} (-1)^k \left[\sum_{|T|=k} S(\mu_T) \right].$$

"$S(\mu_\emptyset) = 1 \log 1 = 0$"

Alternating Sum: Looks kind of like an Euler Char.

More precisely, q-deform $\sum_{|T|=k} S(\mu_T)$:

$$\frac{1}{1-q} \sum_{|T|=k} \mu_T{}^q$$

$\lim_{q \to 0}$ → $\sum_{|T|=k} \#\{ r \in \Omega_T : \mu_T(r) \neq 0 \}$

$\lim_{q \to 1}$ → $\sum_{|T|=k} S(\mu_T)$

From $\sum_{|T|=k} \#\{ r \in \Omega_T : \mu_T(r) \neq 0 \}$:

$I_{q=0} \in \mathbb{Z}$ and we have something that looks like an Euler-Char.

What is this topological Space?

<u>Def:</u> $\mu : \Omega \longrightarrow \mathbb{R}_{\geq 0}$

$$\operatorname{Supp} \mu = \{ \omega \in \Omega : \mu_\omega > 0 \}$$

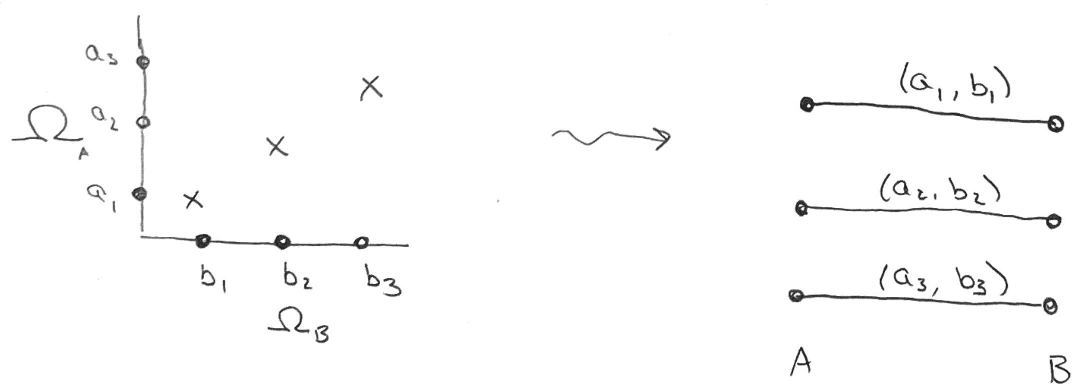
<u>Key Lemma:</u> $\operatorname{Supp} \mu_{AB} \subseteq \operatorname{Supp}(\mu_A \times \mu_B) \cong \operatorname{Supp} \mu_A \times \operatorname{Supp} \mu_B$

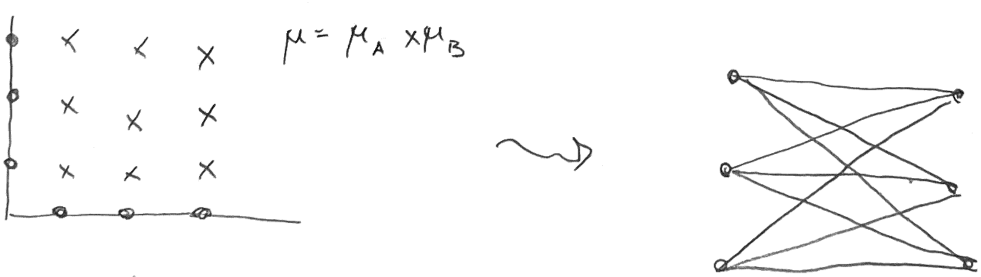
<u>Construction of Our Space:</u>

Bipartite Example: $\mu_{AB} : \Omega_A \times \Omega_B \longrightarrow \mathbb{R}_{\geq 0}$

- Draw a (labelled) point for each element of $\operatorname{Supp} \mu_A \amalg \operatorname{Supp} \mu_B$
- For each $(a,b) \in \operatorname{Supp} \mu_{AB}$ draw an interval w/ ∂ given by $a \in \operatorname{Supp} \mu_A$, $b \in \operatorname{Supp} \mu_B$

<u>Ex:</u>





Complete bipartite graph