Dr. Strangeduality or: how I learned to Stop dozing off and love (the) Boolean Algebras

ASU Math
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v2
Diff Geom Seminar

Prelude

There's a common duality theme in math

$$\text{Spaces} \underset{\text{"Spectrum"}}{\overset{\text{Ring of Functions}}{\rightleftarrows}} (\text{Comm Algebras})^{\text{"op"}}$$

Why the "op"?

$$S \xmapsto{\text{Funs.}} \text{Fun}(S)$$

$$f: S \to R \longmapsto f^*: \text{Fun}(R) \longrightarrow \text{Fun}(S)$$

Examples

- Gelfand Duality

$$\begin{matrix}\text{Compact} \\ \text{Hausdorff}\end{matrix} \underset{\text{Gelfand Spec}}{\overset{\text{Cont. } \mathbb{C}\text{-Valued Funs.}}{\rightleftarrows}} \text{Commutative (unital) } C^*\text{-algebras}^{op}$$

Gelfand Spec = $\mathbb{C}$-Valued homomorphisms (Characters)

- Algebraic Geom:

$$\text{Affine } \mathbb{C}\text{-Varieties} \underset{\text{MaxSpec = max'l ideals}}{\overset{\text{Global Funcs}}{\rightleftarrows}} \text{"Polynomial Rings"}^{op}$$

$$\cap\!|$$

$$\text{Affine Schemes} \underset{\text{Spec = prime ideals}}{\rightleftarrows} (\text{Com. Rings})^{op}$$

Finitely Gen., nilpotent-Free noetherian $\mathbb{C}$-algebras

- Stone Duality

$$\text{Stone} \underset{\text{Stone-Spec}}{\overset{\text{Clopen Sets / } \mathbb{Z}_2\text{-Valued Cont Funcs}}{\rightleftarrows}} (\text{Boolean Algebras / Rings})^{op}$$

Stone = Totally disconnected Compact Hausdorff

Connected Comp are one-point Sets.

Ex

- Discrete Top Spaces
- $\mathbb{Q} \subseteq \mathbb{R}$
- Cantor Set $\subseteq [0,1]$

Recollections on Categories

Moral: A Category is a way of encapsulating a class of mathematical objects and structure preserving morphisms between them.

Ex

- Sets and Functions
- Vector Spaces and linear maps
- Topological Spaces and Cont. Functions

Quasi-Defs

A) A Category $\mathcal{C}$ consists of

1) A collection of "objects" $ob(\mathcal{C})$

2) For each $X, Y \in ob(\mathcal{C})$ a set $\mathcal{C}(X,Y)$ of morphisms / "arrows" $f\ X \longrightarrow Y$

3) A way of composing morphisms when appropriate

$$\left(\begin{array}{rcl} \circ : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) & \longrightarrow & \mathcal{C}(X,Z) \\ (g, f) & \longmapsto & g \circ f \end{array} \right)$$

B) A Functor is a map of categories: $F : \mathcal{C} \longrightarrow \mathcal{D}$ is defined by

1) For each $X \in ob(\mathcal{C})$ an object $FX \in ob(\mathcal{D})$

2) For each $f\ X \longrightarrow Y \in \mathcal{C}(X,Y)$ a morphism $Ff : FX \longrightarrow FY \in \mathcal{D}(FX, FY)$

Examples

- Identity Functor
- "Forgetful Functors": $F : Top \longrightarrow Set$, $Ring \longrightarrow Set, \ldots$
- "Hom" Functors:

$$\mathcal{C}(-,X) : \mathcal{C}^{op} \longrightarrow Set \qquad \mathcal{C}^{op}(X,Y) \mathrel{/\!/}$$

$$Y \longmapsto \mathcal{C}(Y,X)$$

$$f\ Y \longrightarrow Z \longmapsto f^* := (-) \circ f \quad \mathcal{C}(Z,X) \longrightarrow \mathcal{C}(Y,X).$$

Equivalence of Categories

Naively: $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ is an isomorphism of Categories if

$$F \circ G = 1_{\mathcal{D}} \quad \& \quad G \circ F = 1_{\mathcal{C}}$$

Philosophy. One only cares about objects in Categories up to "iso":

Ex.

$$\begin{tikzcd} a \arrow[loop above, "1_a"] \arrow[r, bend left, "f"] & b \arrow[loop above, "1_b"] \arrow[l, bend left, "f^{-1}"] \end{tikzcd} \cong \begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$$

$\rightsquigarrow$

Equivalence of Categories: $F \circ G \xrightarrow[\sim]{\varepsilon} 1_{\mathcal{D}}$

$G \circ F \xrightarrow[\sim]{\eta} 1_{\mathcal{C}}$

$\downarrow$ objects

$\varepsilon_d : F \circ G(d) \xrightarrow{\sim} d \in \mathcal{C}(FGd, d)$

$\eta_c : FG(c) \xrightarrow{\sim} c \in \mathcal{C}(FGc, c)$.

Two Categories $\mathcal{C}$, $\mathcal{D}$ are "dual" if $\mathcal{C}^{op} \simeq \mathcal{D}$.

Want to Show: $\mathrm{Bool}^{op} \simeq \mathrm{Stone}$. ← Totally disconn. Compact Hausdorff + Cont maps

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Boolean Algebras and Boolean Rings

Motivation. The power Set PX of a Set X is more than just a Set: it is equipped w/

- $\vee := \cup : PX \times PX \longrightarrow PX$
- $\wedge := \cap : PX \times PX \longrightarrow PX$
- Complementation: $\neg : PX \longrightarrow PX$

+ a partial order $U \leq V \iff U \subseteq V$.

Quasi-Def

- A Boolean algebra is a poset equipped w/ $\vee$, $\wedge$, and $\neg$ satisfying axioms (mostly deduced by looking at PX)

- A morphism of Boolean algebras is an order-preserving map that preserves $\vee$, $\wedge$, and $\neg$.

(If Boolean Algebras bore you, maybe you can think of them as :)

Def

A Boolean Ring is a ring whose elements are idempotents. ← Think "projections"

Claim

A Boolean Ring is an alternate way of presenting the data of a Boolean Algebra.

On board before talk:

Reverse order in talk

Boolean Ring		Boolean Alg
$xy = y$	$\longleftrightarrow$	$x \le y$
$x \cdot y$	$\longleftrightarrow$	$x \wedge y$
$1 - x$	$\longleftrightarrow$	$\neg x$
$x + y$	$\longleftrightarrow$	$XOR(x,y) = (x \vee y) \wedge \neg(x \wedge y)$
$x + y + xy$	$\longleftrightarrow$	$x \vee y$

Claim: • The additive part of a Boolean Ring is a $\mathbb{Z}_2$-Vector Space

PF: $(x+1)^2 = x \Rightarrow 2x = 0 \Rightarrow \mathbb{Z}_2$-module

(Alt. For Finite Case: Classification of Finite ab. groups)

• A Boolean Ring is Commutative: $(x+y)^2 = x+y \Rightarrow xy + yx = 0.$

<u>Corollary</u> B a Finite Boolean Alg : $|B| = 2^n$ For Some n

<u>Conjecture</u>: • Every Finite B is isomorphic to $PX = \mathrm{FinSet}(X, \{0,1\})$ For Some Finite Set X.

Stronger

• Baby Stone Duality:

$$\begin{tikzcd} \mathrm{FinBool}^{op} \arrow[r, shift left, "?"] \arrow[d] & \mathrm{FinSet} \arrow[l, shift left, "\sim", "\mathrm{FinSet}(-,\{0,1\})"'] \arrow[d, bend left] & \simeq \ \mathrm{FinStone} \\ \mathrm{Bool}^{op} \arrow[r, shift left, "?"] & \mathrm{Stone} \arrow[l, shift left, "\mathrm{Stone}(X,\{0,1\})"] & \end{tikzcd}$$

using discrete topology

<u>Stone Duality</u> $(*)$ $\mathrm{Bool}^{op} \rightleftarrows \mathrm{Stone}$

<u>Recall</u>: We want $(*)$

To construct the right Functors, we make a ridiculous Observation

<u>Observation</u>: The two element Set $\{0,1\}$ has the structure of

• A boolean ring : $\underline{2} := \mathbb{Z}_2$

• A Stone Space : $\overline{2} = \{0,1\}$ + discrete top

So we can look at the Functors

1) $\mathrm{Bool}(-, \underline{2}) : \mathrm{Bool}^{op} \longrightarrow \mathrm{Set}$
"Truth assignments"

2) $\mathrm{Stone}(-, \overline{2}) : \mathrm{Stone} \longrightarrow \mathrm{Set}^{op}$
"Clopen Sets"

$$\left\{ \begin{array}{l} \mathrm{Stone}(X, \overline{2}) \longrightarrow \mathrm{Clopen} \\ \quad f \longmapsto f^{-1}(1) \\ \mathrm{Clopen} \longrightarrow \mathrm{Stone}(X, \overline{2}) \\ \quad A \longmapsto x \longmapsto \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \end{array} \right.$$

<u>Claim</u>:

Observation $\Rightarrow$ There are lifts

L1)

$$\begin{tikzcd}
& \mathrm{Bool}^{op} \arrow[dr, "\mathrm{Forget}"] & \\
\mathrm{Stone} \arrow[ur, dashed, "\mathrm{Stone}(-, \overline{2})"] \arrow[rr] & & \mathrm{Set}^{op}
\end{tikzcd}$$

L2)

$$\begin{tikzcd}
& \mathrm{Stone} \arrow[dr, "\mathrm{Forget}"] & \\
\mathrm{Bool}^{op} \arrow[ur, "\overline{\mathrm{Bool}(-, \underline{2})}"] \arrow[rr, "\mathrm{Bool}(-, \underline{2})"] & & \mathrm{Set}
\end{tikzcd}$$

L1 is easy: • $\overline{2}$ is also a Boolean ring $\underline{2}$

• Functions valued in a ring form a ring under pointwise addition / mult. $\Leftarrow$ ring is also obviously boolean.

For L2:

• Note: $\iota \quad \mathrm{Bool}(B, \underline{2}) \underset{\text{incl}}{\overset{\text{set}}{\hookrightarrow}} \underline{2}^X := \prod_X \underline{2}$ $\leftarrow$ Compact, Haus., Totally discon. $\Leftarrow$ Tychonoff's Thm.

• <u>Prop</u>: Image of ι is closed

• <u>Cor</u>: $\mathrm{Bool}(B, \underline{2})$ is Compact, Haus., Tot disc.

Pf of <u>Prop</u>: (Full version in V1):

- Observe $\mathrm{Bool}(B, \underline{2}) = \overbrace{(\text{Additive homs } B \to \underline{2})}^{A} \cap \overbrace{(\text{mult. homs } B \to \underline{2})}^{B}$
- Show A is closed in $\underline{2}^X$ by using the fact that the diagonal $\Delta_Y \subseteq Y \times Y$ is closed for Y Hausdorff
- Repeat argument for B.

Now we have two Functors

$$\underset{\substack{\shortparallel \\ \mathrm{StSpec}}}{\overline{\mathrm{Bool}}(-,\underline{2})} : \mathrm{Bool}^{op} \rightleftarrows \mathrm{Stone} : \underset{\substack{\shortparallel \\ \mathrm{Clopen}}}{\underline{\mathrm{Stone}}(-,\overline{2})}$$

We want iso's:

$$\varepsilon : I_{\mathrm{Bool}^{op}} \xrightarrow{\sim} \mathrm{Clopen} \circ \mathrm{StSpec}$$

$$\eta : I_{\mathrm{Stone}} \xrightarrow{\sim} \mathrm{StSpec} \circ \mathrm{Clopen}$$

i.e., we want iso's

$$\varepsilon_B : B \xrightarrow{\sim} \underline{\mathrm{Stone}}[\overline{\mathrm{Bool}}[B,\underline{2}], \overline{2}] \quad \text{boolean iso } \forall B$$

$$\eta_S : S \xrightarrow{\sim} \overline{\mathrm{Bool}}[\underline{\mathrm{Stone}}[S,\overline{2}], \underline{2}] \quad \text{continuous map } \forall S$$

Obvious guess: evaluation maps:

$$\varepsilon_B : b \longmapsto (\mathrm{eval}_b : f \longmapsto f(b))$$

This works!

1) eval_b is continuous: it is the restriction of the projection map $\overline{2}^{\mathrm{Bool}[B,2]} \longrightarrow \overline{2}$

2) ε_B is a boolean morphism: $\mathrm{eval}_{b+c} f = \mathrm{eval}_b f + \mathrm{eval}_c f$
$(\mathrm{eval}_{b\cdot c} f = \mathrm{eval}_b f \cdot \mathrm{eval}_c f)$
as f is a Boolean mor

3) ε_B is injective:

$\mathrm{eval}_b \equiv 0 \iff b = 0$ as $\forall b \neq 0$ we have a map

$$f_b : B \longrightarrow 2$$

$$x \longmapsto \begin{cases} 0 & \text{if } x \leq \neg b \quad (x(1-b) = x) \\ 1 & \text{otherwise} \end{cases}$$

$x \in (1-b)B$

4) ε_B is Surjective: Let $\beta \in \underline{\mathrm{Stone}}[\overline{\mathrm{Bool}}[B,\underline{2}], \overline{2}]$

Then $\beta^{-1}(1)$ is open & Closed

open $\Rightarrow \exists (b_i)_{i\in I}$ s.t. $\beta^{-1}(1) = \bigcup_{i\in I} U_{b_i}$

where $U_{b_i} = \{ f \in \overline{\mathrm{Bool}}[B,\underline{2}] ; \mathrm{eval}_{b_i} f = 1 \}$

Closed $\Rightarrow$ Compact ($\overline{\mathrm{Bool}}[B,\underline{2}]$ is Compact)

$\Rightarrow \exists$ a Finite Subcover: $\beta^{-1}(1) = U_{b_{i_1}} \cup \cdots U_{b_{i_n}}$

$\Rightarrow \beta = \mathrm{eval}_{b_{i_1} + \cdots + b_{i_n}}$.

<u>Remark</u>: • If B is a Finite Boolean alg., then $\mathrm{Bool}[B,\underline{2}]$ is the Finite Set "X" of our Conjecture. ε_B is the isomorphism $B \xrightarrow{\sim} PX$.

• One can also prove $\eta_S : S \longmapsto \mathrm{eval}_S$ defines an iso. using Similar arguments.

<u>Further Results</u>

• One can use the Stone-Spectrum Functor $\overline{\mathrm{Bool}}[B,\underline{2}]$ to construct a duality

$$(\sigma\text{-Algebras})^{op} \cong (\text{Measure Spaces})$$

(Loomis-Sikorski Thm.)

• Gelfand Duality Follows mostly by replacing $2 \rightsquigarrow \mathbb{C}$

Caveat: $\mathbb{C} \notin \mathrm{KHaus}$, (but is an elt of Haus).

• Amazingly, there is also an equivalence of Categories

$$\text{Smooth Man.} \underset{\mathbb{R}\text{-Spec}}{\overset{C^\infty(-)}{\rightleftarrows}} (\mathbb{R}\text{-Algebras})^{op}$$

$C^\infty(-)$ = Smooth Man$(-,\mathbb{R})$

$\mathbb{R}$-Spec = $\mathbb{R}$-Alg$(-,\mathbb{R})$