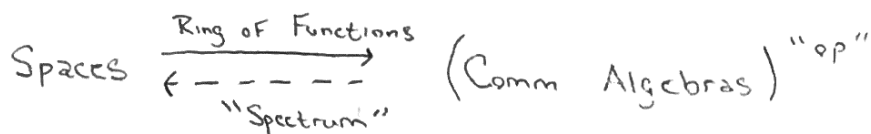


Dr. Strangeduality or: how I learned to Stop dazing off and love (the) Boolean Algebras

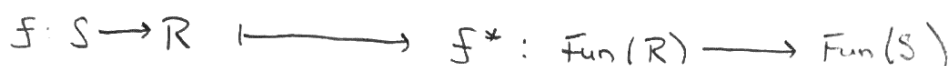
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ASU Math
11/4/2016
v2
D EE Geom
Seminar

Prelude

There's a common duality theme in math

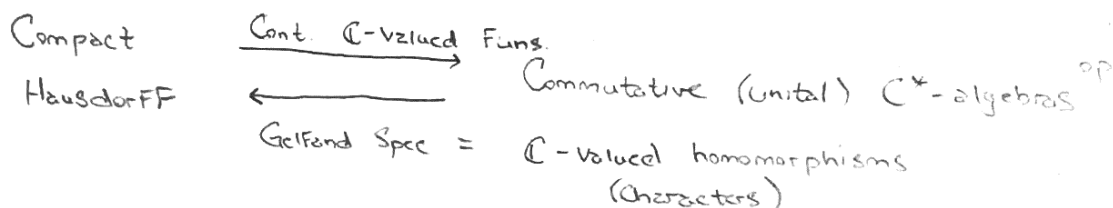


Why the "op"?

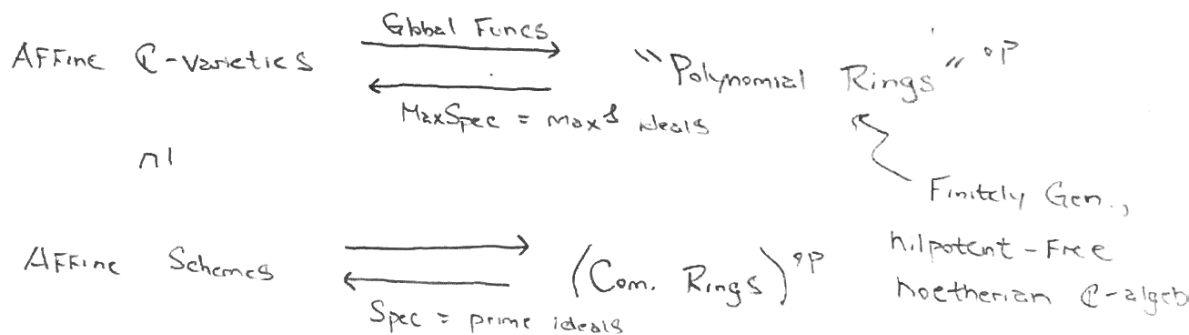


Examples

• Gelfand Duality

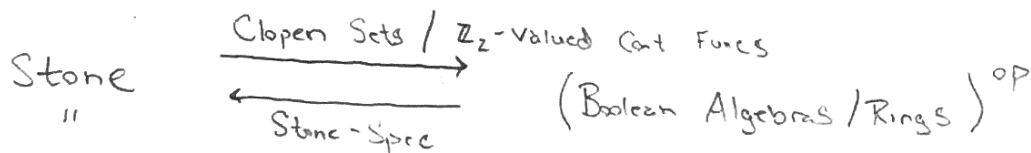


• Algebraic Geom:



• Stone Duality

Connected Comp are
one-point
Sets.



Totally disconnected
Compact Hausdorff

- Ex.
- Discrete Top Spaces
 - $\mathbb{Q} \subseteq \mathbb{R}$
 - Cantor Set $\subseteq [0, 1]$

Recollections on Categories

Moral: A Category is a way of encapsulating a class of mathematical objects and structure preserving morphisms between them.

Ex:

- Sets and Functions
- Vector Spaces and linear maps
- Topological Spaces and Cont. Functions

Quasi-Defs

A) A Category \mathcal{C} consists of

1) A collection of "objects" $\text{ob}(\mathcal{C})$

2) For each $X, Y \in \text{ob}(\mathcal{C})$ a set $\mathcal{C}(X, Y)$ of morphisms / "arrows"
 $f: X \rightarrow Y$

3) A way of composing morphisms when appropriate

$$\left(\begin{array}{l} \circ: \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z) \\ (g, f) \longmapsto g \circ f \end{array} \right)$$

B) A Functor is a map of categories: $F: \mathcal{C} \rightarrow \mathcal{D}$ is defined by

1) For each $X \in \text{ob}(\mathcal{C})$ an object $F X \in \text{ob}(\mathcal{D})$

2) For each $f: X \rightarrow Y \in \mathcal{C}(X, Y)$ a morphism

$$F f: F X \rightarrow F Y \in \mathcal{D}(F X, F Y)$$

Examples

• Identity Functor

• "Forgetful Functors": $F: \text{Top} \rightarrow \text{Set}$, $\text{Ring} \rightarrow \text{Set}$, ...

• "Hom" Functors:

$$\mathcal{C}(-, X): \mathcal{C}^{\text{op}} \longrightarrow \text{Set} \quad \begin{array}{l} e^{\text{op}}(X, Y) \\ \\ Y \longmapsto \mathcal{C}(Y, X) \end{array}$$

$$f: Y \rightarrow Z \longmapsto f^* := (-) \circ f: \mathcal{C}(Z, X) \longrightarrow \mathcal{C}(Y, X)$$

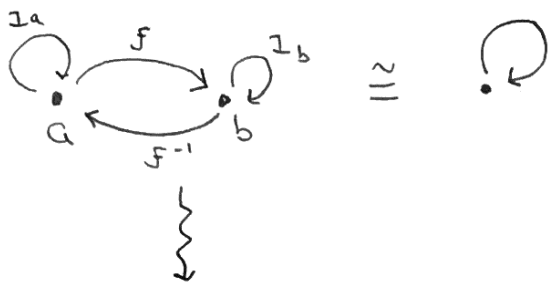
Equivalence of Categories

Naively: $C \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} D$ is an isomorphism of Categories iF

$$F \circ G = I_D \quad \& \quad G \circ F = I_C$$

Philosophy: One only cares about objects in Categories up to "iso":

Ex.



Equivalence of Categories:

$$\begin{matrix} F \circ G \xrightarrow{\eta} I_D \\ G \circ F \xrightarrow{\epsilon} I_C \end{matrix}$$

↓ objects

$$\epsilon_d: F \circ G(d) \xrightarrow{\sim} d \in \mathcal{C}(FGd, d)$$

$$\eta_c: FG(c) \xrightarrow{\sim} c \in \mathcal{C}(FGc, c)$$

Two Categories C, D are "dual" iF $C^{\text{op}} \cong D$.

Want to show: $\text{Bool}^{\text{op}} \cong \text{Stone}$ ← Totally disconn.

Compact Hausdorff
+ Cont maps

Boolean Algebras and Boolean Rings

Motivation: The power Set PX of a set X is more than just a Set:
it is equipped w/

- $V := \cup: PX \times PX \longrightarrow PX$

- $\wedge := \cap: PX \times PX \longrightarrow PX$

- Complementation: $\neg: PX \longrightarrow PX$

+ a partial order $U \leq V \iff U \subseteq V$.

Quasi-Def

- A Boolean algebra is a poset equipped w/ $\vee, \wedge,$ and \neg satisfying axioms (mostly deduced by looking at PX)
- A morphism of Boolean algebras is an order-preserving map that preserves $\vee, \wedge,$ and \neg .

(If Boolean Algebras bore you, maybe you can think of them as:)

Def

A Boolean Ring is a ring whose elements are idempotents. ← Think "projections"

Claim

A Boolean Ring is an alternate way of presenting the data of a Boolean Algebra.

On board before talk:

<u>Boolean Ring</u>		<u>Boolean Alg</u>
$x \cdot y = y$	\longleftrightarrow	$x \leq y$
$x \cdot y$	\longleftrightarrow	$x \wedge y$
$1 - x$	\longleftrightarrow	$\neg x$
$x + y$	\longleftrightarrow	$XOR(x, y) = (x \vee y) \wedge \neg(x \wedge y)$
$x + y + xy$	\longleftrightarrow	$x \vee y$

Reverse order in talk



Claim: • The additive part of a Boolean Ring is a \mathbb{Z}_2 -vector space

PF: $(x+1)^2 = x \Rightarrow 2x = 0 \Rightarrow \mathbb{Z}_2$ -module

(Alt. For finite case: Classification of finite ab. groups)

• A Boolean Ring is commutative: $(x+y)^2 = x+y \Rightarrow xy + yx = 0$.

Corollary B a Finite Boolean Alg. $|B| = 2^n$ For some n

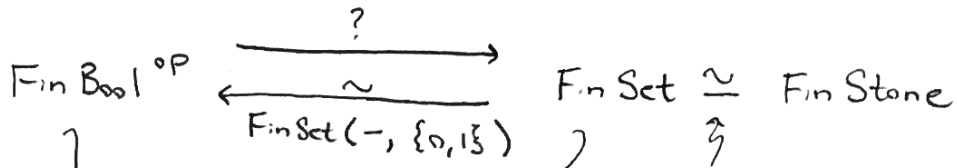
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Conjecture: • Every Finite B is isomorphic to $\mathcal{P}X = \text{FinSet}(X, \{0,1\})$

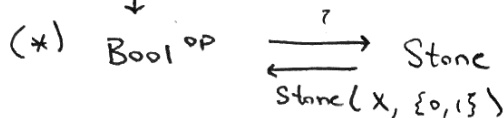
For some Finite Set X .

Stronger

• Baby Stone Duality:



Stone Duality



Recall: We want $(*)$

To construct the right functors, we make a ridiculous observation

Observation: The two element set $\{0,1\}$ has the structure of

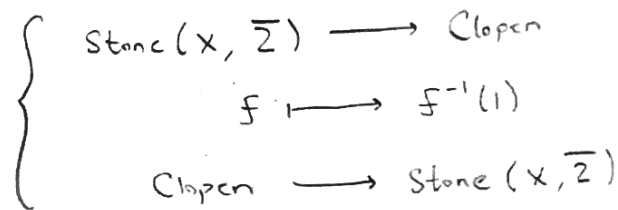
- A boolean ring: $\mathbb{Z} := \mathbb{Z}_2$
- A Stone Space: $\overline{\mathbb{Z}} = \{0,1\} + \text{discrete top.}$

So we can look at the functors

1) $\text{Bool}(-, \mathbb{Z}) : \text{Bool}^{\text{op}} \longrightarrow \text{Set}$
 "Truth assignments"

2) $\text{Stone}(-, \overline{\mathbb{Z}}) : \text{Stone} \longrightarrow \text{Set}^{\text{op}}$

"Clopen Sets"

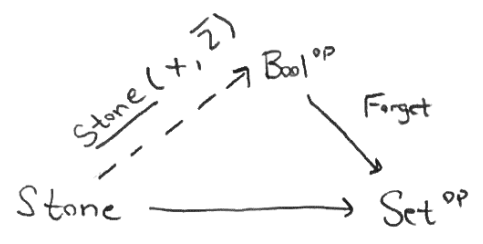


$$A \longmapsto x \longmapsto \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

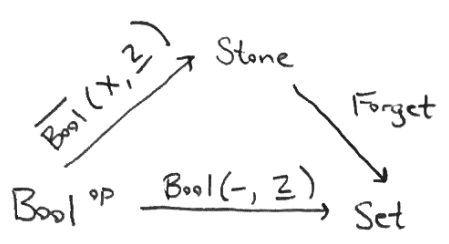
Claim:

Observation \Rightarrow There are lifts

L1)



L2)



L1 is easy: \bullet \underline{Z} is also a Boolean ring \underline{Z}

\bullet Functions valued in a ring form a ring under pointwise addition/mult.

\leftarrow ring is also obviously boolean.

For L2:

\bullet Note: $\mathcal{L} \text{ Bool}(B, \underline{Z}) \xrightarrow[\text{incl}]{\text{set}} \underline{Z}^X := \prod_X \underline{Z}$

\bullet Prop: Image of \mathcal{L} is closed

\bullet Cor: $\text{Bool}(B, \underline{Z})$ is Compact, Haus., Tot disc.

Compact, Haus.,
 \Downarrow
 Tychonoff's Thm. Totally discn.

PF of Prop: (Full version in v1):

- Observe $\text{Bool}(B, \underline{Z}) = (\text{Additive homs } B \xrightarrow{A} \underline{Z}) \cap (\text{mult. homs } B \xrightarrow{B} \underline{Z})$
- Show A is closed in \underline{Z}^X by using the fact that the diagonal $\Delta_Y \subseteq Y \times Y$ is closed for Y Hausdorff
- Repeat argument for B .

Now we have two Functors

$$\overline{\text{Bool}}(-, \underline{2}) : \text{Bool}^{\text{op}} \begin{matrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{matrix} \text{Stone} : \underline{\text{Stone}}(-, \overline{2})$$

\Downarrow StSpec \Downarrow Clopen

We want iso's:

$$\varepsilon : \mathbb{I}_{\text{Bool}^{\text{op}}} \xrightarrow{\sim} \text{Clopen} \circ \text{StSpec}$$

$$\eta : \mathbb{I}_{\text{Stone}} \xrightarrow{\sim} \text{StSpec} \circ \text{Clopen}$$

i.e., we want iso's

$$\varepsilon_B : B \xrightarrow{\sim} \underline{\text{Stone}}[\overline{\text{Bool}}[B, \underline{2}], \overline{2}] \rightsquigarrow \text{boolean iso. } \forall B$$

$$\eta_S : S \xrightarrow{\sim} \overline{\text{Bool}}[\underline{\text{Stone}}[S, \overline{2}], \underline{2}] \rightsquigarrow \text{Continuous map } \forall S$$

Obvious guess: evaluation maps:

$$\varepsilon_B : b \longmapsto (\text{eval}_b : f \longmapsto f(b))$$

This works!

1) eval_b is Continuous: it is the restriction of the projection map $\overline{2} \text{Bool}[B, \underline{2}] \longrightarrow \overline{2}$

2) ε_B is a boolean morphism: $\text{eval}_{b+c} f = \text{eval}_b f + \text{eval}_c f$
 $(\text{eval}_{b \cdot c} f = \text{eval}_b f \cdot \text{eval}_c f)$
 as f is a Boolean mor

3) ε_B is injective:

$\text{eval}_b \equiv 0 \Leftrightarrow b = 0$ as $\forall b \neq 0$ we have a map

$$f_b : B \longrightarrow \overline{2}$$

$$x \longmapsto \begin{cases} 0 & \text{if } x \leq b \text{ (} x(1-b) = x \text{)} \\ 1 & \text{otherwise} \end{cases}$$

$\hat{=} \begin{matrix} x \in (1-b)B \\ \hat{=} \end{matrix}$

4) ε_B is surjective: Let $\beta \in \text{Stone}[\overline{\text{Bool}}[B, \underline{z}], \overline{z}]$

Then $\beta^{-1}(1)$ is open & closed

open $\Rightarrow \exists (b_i)_{i \in I}$ s.t. $\beta^{-1}(1) = \bigcup_{i \in I} U_{b_i}$

where $U_{b_i} = \{f \in \overline{\text{Bool}}[B, \underline{z}] ; \text{eval}_{b_i} f = 1\}$

Closed \Rightarrow Compact ($\overline{\text{Bool}}[B, \underline{z}]$ is compact)

$\Rightarrow \exists$ a finite subcover: $\beta^{-1}(1) = U_{b_{i_1}} \cup \dots \cup U_{b_{i_n}}$

$\Rightarrow \beta = \text{eval}_{b_{i_1} + \dots + b_{i_n}}$

Remark: • IF B is a finite Boolean alg., then $\text{Bool}[B, \underline{z}]$ is the finite set "X" of our Conjecture. ε_B is the isomorphism $B \xrightarrow{\sim} PX$.

- One can also prove $\eta_S : S \rightarrow \text{eval}_S$ defines an iso. using similar arguments.

Further Results

$\overline{\text{Bool}}[B, \underline{z}]$

- One can use the Stone-Spectrum Functor to construct a duality

$$(\sigma\text{-Algebras})^{\text{op}} \cong (\text{Measure Spaces})$$

(Loomis-Sikorski Thm.)

- Gelfand Duality follows mostly by replacing $\mathbb{Z} \rightsquigarrow \mathbb{C}$
 Caution: $\mathbb{C} \notin \text{KHaus}$, (but is an elt of Haus).

- Amazingly, there is also an equivalence of categories

$$\begin{array}{ccc} \text{Smooth Man.} & \xrightarrow{C^\infty(-)} & (\mathbb{R}\text{-Algebras})^{\text{op}} \\ & \xleftarrow{\mathbb{R}\text{-Spec}} & \\ & & \text{= Smooth Man}(-, \mathbb{R}) \\ & & \text{= } \mathbb{R}\text{-Alg}(-, \mathbb{R}) \end{array}$$