Morse (t) I listen to this talk? ASU 11/6/2015

Recall: Morse Theory

(Analysis)
Gradient Flow of a (Morse) Function $f: M \longrightarrow \mathbb{R}$ ⇝ (topology) Cell decomposition of M

Morse-Smale Homology ↓ Cellular homology ↓

$$(C_\bullet^{MS}, \partial^{MS}) \cong (C_\bullet^{CW}, \partial^{CW})$$

(Involves a direct translation of the analytic data of Flow lines into "algebraic data")

What is the Morse-Smale Complex?

Quick Defs

Setup: M smooth, cpt, n-dim'l
$f: M \longrightarrow \mathbb{R}$

Def

- $Crit(f) := \{ \ell \in M : df_\ell = 0 \}$
- f is **Morse** if $\forall c \in Crit(f)$; $Hess_c(f) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right)^n_{i,j=1} \Big|_{x=c}$ is non-degen.
- $c \in Crit(f)$; $Ind(c) := \#$ neg e'values of $Hess_c(f)$

Equip M w/ a metric g and study the negative gradient flow of f

$$\Psi_s : M \longrightarrow M = \text{time } s \text{ Flow of } -\nabla f$$

Q: Local behaviour @ crit points?

Def: $p \in \mathrm{Crit}(f)$

$$\mathcal{U}(p) := \left\{ m \in M : \lim_{s \to -\infty} \Psi_s(m) = p \right\} \leftarrow \text{(unstable)}$$

$$S(p) := \left\{ m \in M : \lim_{s \to +\infty} \Psi_s(m) = p \right\} \leftarrow \text{(stable)}$$

Local picture around p: Use Morse Lemma.

$\exists$ local coords $(x^i)_{i=1}^n$ s.t.

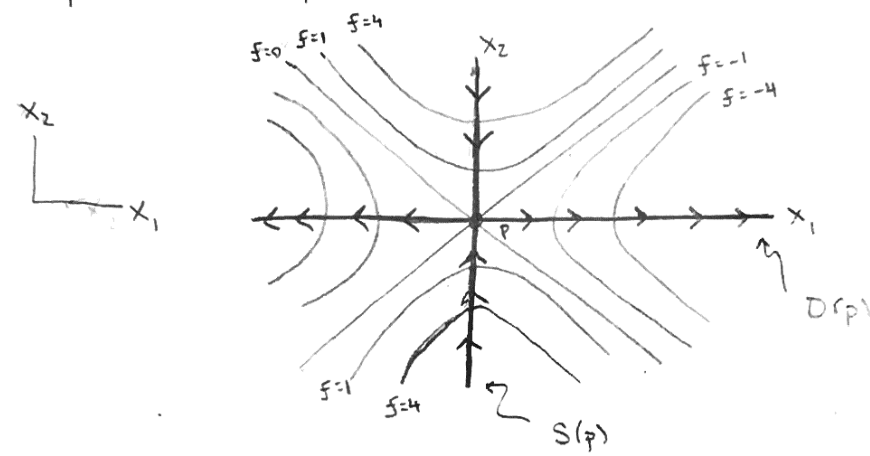
$$f = f(p) + -x_1^2 - \cdots - x_{\mathrm{ind}(p)}^2 + x_{\mathrm{ind}(p)+1}^2 + \cdots + x_n^2$$

$\Rightarrow$

$$\dim \mathcal{U}(p) = \mathrm{ind}(p)$$

$$\dim S(p) = n - \mathrm{ind}(p)$$

Eg. $f = -x_1^2 + x_2^2$



Now define

$$\mathcal{M}(p,q) := \mathcal{U}(p) \cap S(q) \Big/ \underset{\uparrow}{\mathbb{R}} = \left\{ \text{Flow lines } p \to q \right\} \Big/ \left(\text{Time Translation} \right)$$

Translation by
$\Psi : \mathbb{R} \to \mathrm{DIFF}(M)$

So (assuming Morse-Smale $\Rightarrow \mathcal{U}(p) \pitchfork S(p)$)

$$\boxed{\dim \mathcal{M}(p,q) = \mathrm{ind}(p) - \mathrm{ind}(q) - 1}$$

Interested in when $\dim \mathcal{M}(p,q) = 0$! $\quad (\mathrm{ind}(q) = \mathrm{ind}(p) - 1)$

Def Morse-Smale Complex

$$C_i^{MS}(M,g,f) := \mathbb{Z}\,\mathrm{Crit}_i(f) \quad \text{index } i \text{ critical points}$$

$$\partial^{MS} p = \sum_{q \in \mathrm{Crit}_{i-1}} \#\mathcal{M}(p,q) \cdot q$$

Signed # (Choose orientations for unstable manifolds)

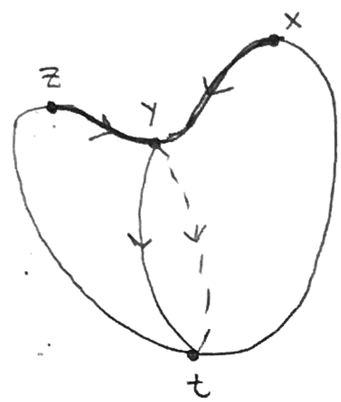
Q1: Is $\#\mathcal{M}(p,q)$ Finite?

Q2: $(\partial^{MS})^2 = 0$?

A: $\mathcal{M}(p,q)$ can be compactified (for any p,q) by adding in "broken flow lines" $\Rightarrow$ Q1 and Q2 are true

Sum of ∂ points of a compact 1-man $= 0$.

Example



$C_2 \cong \mathbb{Z}\langle x, z\rangle \qquad \partial x = \partial z = y$

$C_1 \cong \mathbb{Z}\langle y\rangle \qquad \partial y = (+1-1)\cdot t = 0$

$C_0 \cong \mathbb{Z}\langle t\rangle$

Remark: Unstable manifolds provide cell decomp. as described in beginning.

1982: Ed Witten: The Morse-Smale complex arises naturally from supersymmetric Quantum Mechanics on M.

Crash Course in QM: Particle on (M,g)

Classical Mech

- Phase Space T^*M
- $H: T^*M \longrightarrow \mathbb{R}$
- Ham. Flow $\Phi: \mathbb{R} \longrightarrow \mathrm{Diff}(M)$

Quantization $\rightsquigarrow$

Quantum Mech

- Hilbert Space $\mathcal{H} = L^2(M)$
- $\hat{H}: \mathcal{H} \longrightarrow \mathcal{H}$ self-adjoint
- $\mathbb{R} \longrightarrow U(\mathcal{H})$

  $t \longmapsto e^{iHt/\hbar}$

QM

Rep. of $\mathbb{R}\langle \hat{H} \rangle$

Susy QM

Rep. of $\mathbb{R}\langle \hat{H}, \hat{Q} \rangle$

$$\hat{H} = \hat{Q}^2$$ "2nd order diff'l op. written as square of a 1st order op.

$$[\hat{H}, \hat{Q}] = 0$$

Ex: Free Particle on (M,g) (compact)

1) **QM**. $\mathcal{H} = \Omega^0(M)$; $\langle f, g \rangle = \int f \wedge *g$ Hodge Star op: $\Omega^k \longrightarrow \Omega^{n-k}$

$\hat{H} = \Delta$ Hodge Laplacian

$$\left(= -g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right)$$

2) **Susy QM**: $\mathcal{H} = \Omega^\bullet(M) \otimes_{\mathbb{R}} \mathbb{C}$

$$\hat{H} = \Delta = (d + d^*)^2 : \Omega^k \longrightarrow \Omega^k$$

$$\hat{Q} = d + d^* : \Omega^k \longrightarrow \Omega^{k+1} \oplus \Omega^{k-1}$$

Natural Question: What is the Spectrum of $\hat{H}$? ("Stationary" States)

Note: $\hat{H} = \hat{Q}^2 \Rightarrow$ Smallest eigenvalue ≥ 0

$$\operatorname{Ker}(\hat{H}) = \{ \omega \in \Omega^\bullet : \Delta\omega = 0 \} = \mathcal{H}^\bullet(M)$$

harmonic Forms on M

Hodge Theorem:

$$\mathcal{H}^\bullet(M) \cong H^\bullet_{dR}(M)$$

(as $\mathbb{Z}$-graded Vector Spaces).

Now Consider an $\mathbb{R}_{\geq 0}$-Family of deformations of $\hat{H}$: "Add a potential".

$$\hat{H}_s = (d_s + d_s^*)^2$$

$$d_s := d + s\,dh \wedge (-) : \Omega^\bullet \longrightarrow \Omega^{\bullet+1}$$

where

- $h : M \longrightarrow \mathbb{R}$ Morse
- $s \in \mathbb{R}_{\geq 0}$

Q: How does $\mathrm{Ker}(\hat{H}_s)$ change as I vary s?

Observation:

$$d_s = e^{sh} \circ d \circ e^{-sh} \rightsquigarrow \text{mult. by } e^{-sh}$$

$\Rightarrow$ The map
$$\Omega^\bullet \longrightarrow \Omega^\bullet$$
$$v \longmapsto e^{sh} v$$

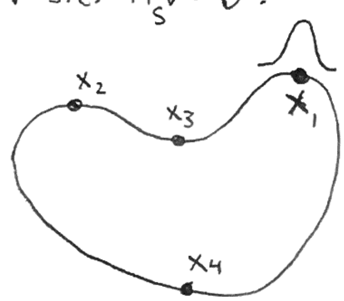
1) Gives an iso. of complexes $(\Omega^\bullet, d_s) \xrightarrow{\sim} (\Omega^\bullet, d)$

2) " " " $\mathrm{Ker}(\hat{H}_s) \xrightarrow{\sim} \mathrm{Ker}(\hat{H})$

$$\Rightarrow \quad \mathrm{Ker}(\hat{H}_s) \simeq H^\bullet_{dR}(M)$$

The $s \to \infty$ limit:

Guess at v s.t. $\hat{H}_s v = 0$:



($\hat{H}_s v = 0$
$\Rightarrow d_s v = 0$
$\Rightarrow v$ localized around $\mathrm{Crit}(h)$).

Claim: $d_s x_i \neq 0$!

Why? "Instanton Corrections"

$\{\ell_1, \ldots, \ell_r, \psi_1, \ldots\}$ orthonormal basis

$$d_s x_i = \sum_{\ell \in \mathrm{Crit}(h)} \langle \ell, d_s x_i \rangle \ell + \underbrace{\sum_{k=1}^{\infty} \langle \psi_k, d_s x_i \rangle \psi_k}_{C}$$

<u>Claim</u> : • $\langle \ell, d_s x_i \rangle$ = Signed Sum over gradient Flow lines $x_i \longrightarrow \ell$ $+ \mathcal{O}(\frac{1}{s})$

• $C = \mathcal{O}(\frac{1}{s})$

Thus,

$$(\Omega^{n-\bullet}, d_s) \xrightarrow[s \to \infty]{} (C^{MS}_{\bullet} \otimes_{\mathbb{Z}} \mathbb{C}, \partial^{MS})$$

and we expect

$$H^{n-\bullet}_{dR}(M) \otimes_{\mathbb{R}} \mathbb{C} \cong H^{MS}_{\bullet}(M) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^{Sing}_{\bullet}(M; \mathbb{C})$$

What are instanton Corrections?

Quantum Mechanics on $M \cong \mathbb{R}$ w/ potential $U : M \longrightarrow \mathbb{R}$

Quantum Hamiltonian:

$$\hat{H} = -\frac{1}{\hbar^2} \frac{\partial^2}{\partial x^2} + U(x)$$

What are its lowest eigenstates?

(See 5 old)

Ex:

$U =$ 

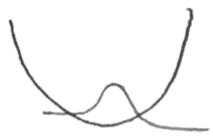
Lowest energy Eigenstates : $\hat{H}\psi = E_{low}\,\psi$

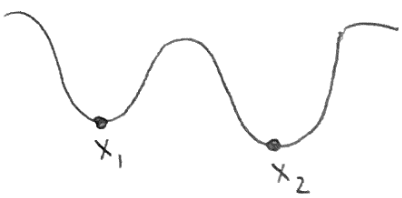
Classically:

$\gamma(t) \equiv$ minimum Point of U

$\rightsquigarrow$

Quantum Mechanically





$\rightsquigarrow$

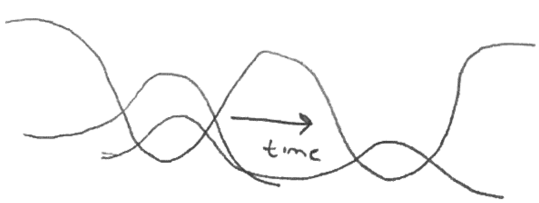
Guess:

$= \psi_1$

and

$= \psi_2$

But in reality : tunneling !



$\psi_1 + \psi_2$ (or $\psi_1 - \psi_2$) is the actual ground state.

Path Integral Point of View: Why is $e^{-iHt}\psi_1 \not\propto \psi_1$?

Answer: Quantum Corrections from "Instantons"