This One Weird Trick has Algebraic Functions Generating DT-Invariants From home! (Kansas State: 11/4/14)

DT-Invariants For Quivers

Main Example:

$$K_m = \underset{q_1}{\circ} \;\overset{\vdots}{\rightrightarrows}\; \underset{q_2}{\circ} \Big\} \; m \geq 1 \text{ arrows}$$

Motivation: Counting Quiver Moduli

Recall: Let $Q = (Q_0, Q_1)$ be a quiver; a rep$\underline{\underline{n}}$ For Q is

- $V = (V_i)_{i \in Q_0}$; V_i a Vector Space
- $T_{i \to j} \in \mathrm{Hom}(V_i, V_j)$ For each $i \to j \in Q_1$

A morphism between V and W is a Collection of:

$$\begin{tikzcd} V_i \arrow[r, "T_{ij}^V"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & V_j \arrow[d] \\ W_i \arrow[r, "T_{ij}^W"] & W_j \end{tikzcd}$$

- Want to Count (Semi)-Stable representations

Def:

- $Z : \mathbb{Z}Q_0 \longrightarrow \mathbb{C}$ a Stability Cond.
- Let $V \in \mathrm{Rep}(Q)$; then V is (Semi)-Stable if

$$\mathrm{Arg}(Z_{\dim(W)}) \underset{(-)}{<} \mathrm{Arg}(Z_{\dim(V)}) \quad ; \quad \dim(V) \in \mathbb{Z}_{\geq 0} Q_0$$

dimension vector

$\forall W \leq V$.

- $\mathcal{M}_Q^{s(s)}(d; Z)$ = moduli space of (Semi)-Stable moduli w/ dim vector $d \in \mathbb{Z}_{\geq 0} Q_0$.

Projective Variety For m-Kronecker Quiver

Rmk: Stab(Q) is a Complex manifold; (locally homeo. to $\mathrm{Hom}(\mathbb{Z}Q_0, \mathbb{C})$)

Want an index that Counts the "#" of (Semi)-Stable moduli that is Constant as we vary Z Continuously in Stab(Q).

Correct Idea:

$\mathbb{Q} \ni \Omega(d; Z) =$ "Weighted Count of Semi-stable objects of dimension $d \in \mathbb{Z}_{\geq 0} Q_0$"

$\Big\downarrow$ Joyce-Song, Behrend

Weighted Euler-Characteristic $\chi^w(\mathcal{M}^{ss}(d))$

$\big(\mathcal{M}^s = \mathcal{M}^{ss} \Rightarrow \chi^w = \pm \chi \big)$

Smooth moduli

Mild Lies:

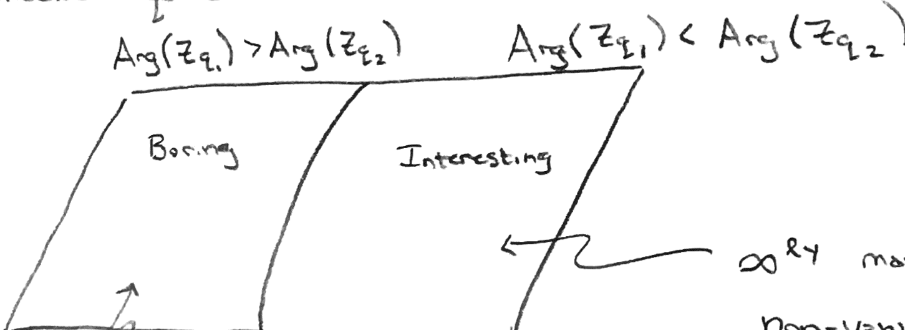
$\Omega(d; Z)$ is piecewise-Constant as we vary $Z \in$ Stab(Q):
$\mathbb{R}$ Codim $=1$ walls where Ω "jumps".

Jumping $\longleftrightarrow$ KS Wall-Crossing Formula

$$\{\Omega(d, Z_0)\}_d \rightsquigarrow \{\Omega(d; Z_1)\}_d$$

$$Z : [0,1] \longrightarrow \mathrm{Stab}(Q)$$

Ex: m-Kronecker quiver:



∞^{ly} many non-vanishing $\Omega(d)$.

$\Omega(q_1) = \Omega(q_2) = +1$; all else vanishing

Key-Player in Wall-Crossing Formula: $d = aq_1 + bq_2$; a, b Coprime

Define

$$T_{a/b} = \prod_{n \geq 1} \left(1 - [(-1)^N t]^n\right)^{n\,\Omega(n \cdot d;\, z)} \in \mathbb{Q}[[t]]$$

(*) <u>Claim</u> : $T_{a/b} \in \overline{\mathbb{Q}(t)}$ (Via "Spectral Networks").

E.g.: $a = b = 1$;

$$T_{a/b} = P^m$$

$$0 = P - tP^{(m-1)^2} - 1$$

$a = 3,\ b = 2$: 39^{th}-degree polynomial $\mathcal{F} \in (\mathbb{Z}[t])[y]$

s.t. $\mathcal{F}(T_{3/2}) = 0$

<u>Remark</u>

Define

$$E_{a/b} = \prod_{n \geq 1} (1 - t^n)^{n\,\chi(\mathcal{M}^s_{K_m}(nd))} \in \mathbb{Z}[[t]]$$

Then Reineke Showed

$$(t \cdot T_{a/b}^{N}) \circ (t\, E_{a/b}^{-N}) = t$$

Inverse Functions!

<u>Corollary to (*)</u> : $E_{a/b} \in \overline{\mathbb{Q}(t)}$

E.g. $a = b = 1$: $E_{1} = (1 - t)^{-m}$

$a/b = 3/2$: $\mathcal{G}(E_{3/2}) = 0$ For $\mathcal{G}$ a $\underline{9}^{\text{th}}$ degree poly.

Corollary to Algebraicity and Integrality: Exponential Growth.

Let $G \in \overline{\mathbb{Q}(t)} \cap \mathbb{Z}[[t]]$ generate $(\beta_n)_{n=1}^{\infty}$:

$$G = \prod_{n \geq 1} \left(1 - (st)^n\right)^{n\beta_n} \quad ; \quad s \in \{\pm 1\}$$

Then

$$\beta_n = s^n C \cdot n^{-2-\alpha} \sum_{i=1}^{r} p_i^{-n} + O\left(n^{-2-\alpha-\varepsilon} R^{-n}\right)$$

For some $\{p_i\}_{i=1}^{r} \subset \overline{\mathbb{Q}}$ s.t. $|p_1| = \cdots = |p_r| = R \leq 1$

$\alpha \in \mathbb{Q}_{\geq 0}$, $\varepsilon > 0$.

DT-invariants in $\mathcal{N}=2$, $d=4$ QFT

$\mathcal{N}=2$ Field theory on $\mathbb{R}^{3,1}$ $\xrightarrow[\text{Physics}]{\text{Low Energy}}$

$\Gamma\left(\begin{smallmatrix} V \\ \downarrow \\ \mathbb{R}^{3,1}\end{smallmatrix}\right)$

- Moduli Space of Vacua $\mathcal{B}$ (∂-Cond. at ∞)
- $(\hat{\Gamma}, \langle \cdot, \cdot \rangle) \to \mathcal{B}$ local-System of Symp. lattices of electric/mag. Charges e.g. $\Gamma \cong \mathbb{Z} \times \mathbb{Z}$
- $Z \in \mathrm{Hom}(\hat{\Gamma}_u, \mathbb{C})$

$\Omega^{BPS}(\gamma; u)$ = Weighted Count of "BPS States" of Charge $\gamma \in \hat{\Gamma}_u$

$= \mathrm{Tr}_{\mathcal{H}_{BPS}(\gamma)}$ (Some operator)

Piecewise Stable as u is Varied.

BPS Quiver

$u \in B \rightsquigarrow$ "BPS Quiver" Q_u

- Nodes: "Primitive Charges $\{\gamma_i\} \subset \hat{\Gamma}_u$ w/ $\Omega(\gamma_i; u) = +1$" (Caveat: Some missing info)
  wrt some basis,
- Arrows: Symplectic pairing

E.g. $\hat{\Gamma}_u = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$; $\langle \gamma_1, \gamma_2 \rangle = m \in \mathbb{Z}_{\geq 1}$

$$\begin{tikzcd} \gamma_1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\vdots"] & \gamma_2 \end{tikzcd} \Big\} m$$

<u>Note</u>: $Z_u \rightsquigarrow$ Stability Cond on Q_u

<u>Claim</u>: $\Omega^{BPS}(\gamma; u) = \Omega^{DT}(\gamma; Z_u)$ (Denef, Gaiotto-Moore-Neitzke, Manschot-Pioline-Sen (in SUGRA))
ss = st

(<u>Rmk</u>: $T = \prod_{n \geq 1} (1 - \ldots)^{n\Omega^{BPS}(n\gamma)}$ is a part. Func. for "Halo" states)

Spectral Networks

Theory of Class $S[A_{K-1}, C, D]$

- C: holomorphic Curve
- D: Divisor
- $K \geq 1$ an integer

$\rightsquigarrow$

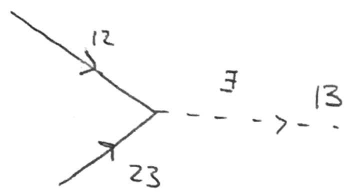
$$\begin{tikzcd} \mathcal{M}_{Higgs}(C,D) \arrow[d, "\pi_{Hitchin}"] \\ B = \{ \text{Spectral Covers } \Sigma_u \to C \} \end{tikzcd}$$

- $\hat{\Gamma}_u = H_1(\Sigma_u; \mathbb{Z})$.

BPS States at Vacuum $u \in B$ (For theory on $\mathbb{R}^{3,1}$) $\rightsquigarrow$ Decorated directed Graphs on C w/ Fixed Vertices at branch points of $\Sigma_u \longrightarrow C$

Decoration: Each edge is labelled by an ordered pair of Sheets of $\Sigma_u \longrightarrow C$

Some "Rules":

1) 

, ∃ 

2) Graph Lifts to a digraph on Σ_u that Supports a non-trivial Closed Cycle on Σ_u. Call it ℓ.

Then

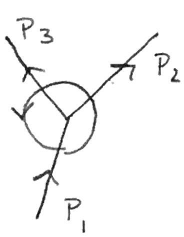
$\ell' \neq \ell \circ \ell \circ \cdots \circ \ell$

$\Omega(n[\ell], u) =$ "#" "Primitive" Closed loops on Lifted graph w/ class $n[\ell] \in H_1(\Sigma_u; \mathbb{Z})$.

Encode Ω via Generating Functions:

$$\rightsquigarrow \quad \mathbb{T}_p = (1 + i\,\Upsilon_p\, \mathbb{E}_{ij})(1 - i\,\Delta_p\, \mathbb{E}_{ji})$$

Υ_p, Δ_p Formal Symbols

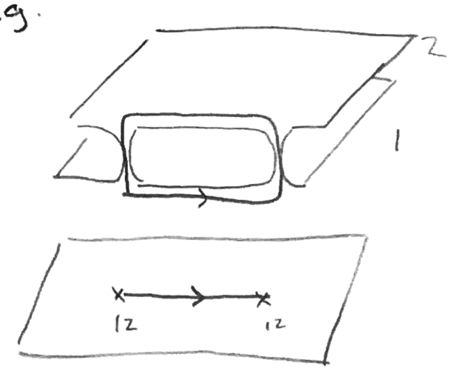


$$\rightsquigarrow \quad \mathbb{T}_{p_2}^{-1}\, \mathbb{T}_{p_3}^{-1}\, \mathbb{T}_{p_1} = \mathbb{1}$$

$$\mathbb{T}_p^{-1} \cdot \mathbb{M} = \mathbb{1}$$

$$\mathbb{M} = t\,\mathbb{E}_{ij} + t^{-1}\,\mathbb{E}_{ji}$$

E.g.



These determine the γ_p and Δ_p as algebraic functions.

$\exists$ sol$^{\underline{n}}$s in $\mathbb{Z}[[t]]$:

$$\gamma_p = \sum_{n=1}^{\infty} a_n t^n \quad ; \quad a_n = \#$$ of loops passing through lift of p w/ agreeing orientation

$$\Delta_p = \sum_{n=1}^{\infty} b_n t^n \quad ; \quad b_n = \text{''} \quad \text{''}$$ disagreeing orientation.

T = Polynomial Combination of γ_p & Δ_p

$$= 1 + \sum_{n=1}^{\infty} \alpha_n t^n \quad ; \quad \alpha_n = \# \text{ of loops } \tilde{\ell} \text{ w/ } [\hat{\ell}] = n[\ell]$$

$$= \prod_{n \geq 1} (1 - (\pm t)^n)^{n\Omega(n[\ell])}$$

Ω: # primitive loops

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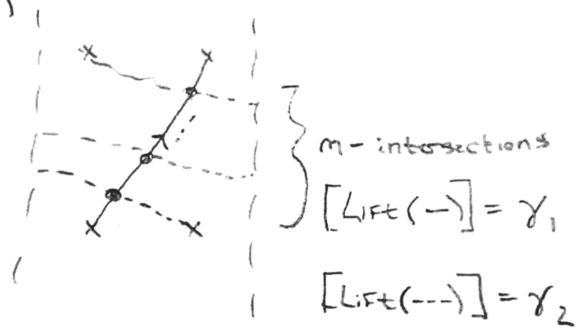
m-Kronecker DT invariants: Interesting stab. Cond$^{\underline{n}}$.

$$\underset{\gamma_1}{\circ} \; \vdots \; \underset{\gamma_2}{\circ} \quad \longleftrightarrow \quad \langle \gamma_2, \gamma_1 \rangle = m \quad \longleftrightarrow$$

(*)



$C = S^1 \times \mathbb{R}$

$(a,b|m)$-herd Deformation of (*)

Eg.: $a = b = 1$, $m = 3$