Functional Equations and DT-invariants From Spectral Networks: Revenge of the m-herds

Punchline:

Degenerate Spectral Networks ⇝ Functional Equations For DT invariants (associated to the Kronecker m-Quiver)

Consequences

- Kronecker m-quiver appears as a Sub-quiver of the BPS quiver For pure SU(3) SYM

Functional Equations ⇒ # BPS States of mass $m_* = n m_0$ $\sim C \cdot n^{-5/2} e^{K \cdot n}$

- The "non-abelianization map" attached to such networks is not a map

$$\psi_N : (\mathbb{C}^\times)^{2g_\Sigma} \longrightarrow \mathrm{Loc}_K(C, \mathbb{D})$$

$$(\mathbb{C}^\times)^{2g_\Sigma} \cong \mathrm{Loc}_1(\Sigma, \mathbb{D})$$

← Local Systems w/ Some asymptotic about Connections as you approach a divisor

but a map

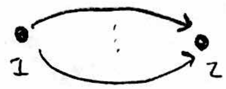
$$\psi_N : (\mathbb{C}^\times)^{2g_\Sigma} \times (\text{Cover of } \mathbb{C}^\times) \longrightarrow \mathrm{Loc}_K(C, \mathbb{D})$$

DT-invariants

(Q an (acyclic) quiver

$\Omega(\gamma) =$ " Virtual # of Semi-Stable objects in $D^b \operatorname{Rep}(Q)$ of class $\gamma \in K_0$ ")

↖ " Invar. under mutation "

$Q =$  $\}$ $m \geq 1$ arrows

↓

Poisson Algebra $\mathbb{C}[[x_1, x_2]]$ (with $x_1 = x$, $x_2 = y$) w/ $\{x, y\} = m\,xy$

Let $F \in \mathbb{C}[[t]]$, $F(0) = 1$, then define auts:

$$T_{(a,b),F} : \begin{cases} x \longmapsto x\,F(x^a y^b)^{-mb} \\ y \longmapsto y\,F(x^a y^b)^{ma} \end{cases}$$

$\gcd(a,b) = 1$

(" Time 1 Flow defined By Hamiltonian $H(x,y) = \exp\{x^a y^b F'(x^a y^b)\}$ "

Translations in the right Coords " $p = x^a y^b$, $q = x^\alpha y^\beta$, $a\alpha - b\beta = 1$. ")

Particularly interested in

$$T_{(a,b)} := T_{(a,b),\,(1 - (-1)^{mab} x^a y^b)}$$

DT-invariants appear as Commutators:

$$T_{(1,0)}\, T_{(0,1)} = \overleftarrow{\prod_{b/a\text{-decreasing}}} \left(T_{(a,b)}\right)^{d(a,b,m)}$$

$$= T_{(0,1)} \left(\cdots\cdots \right) T_{(1,0)}$$

↖ $m = 2$

$T_{(1,0)} T_{(0,1)} = T_{(0,1)} T_{(1,1)}^{-2} T_{(1,0)}$

Claim:

$\exists F \in \mathbb{C}[[t]]$

$$T_{(a,b),F} = \prod_{n=1}^{\infty} T_{(na,nb)}$$

Then

$$F = \prod_{n=1}^{\infty} \left(1 - (-1)^{mab}\, t\right)^{n\,d(na,nb,m)}$$

<u>Reineke</u>: F Satisfies a Functional Eqn depending on $\chi(M^{st}_{(ka,kb)})$:

$$F((-1)^{a+b}t) = G(t)$$

$$G(t) = \prod_{n\geq 1}\left(1-\left(t\,G(t)^{mab-a^2-b^2}\right)^n\right)^{-n\chi(M^{st}_{(na,nb)})}$$

E.g. $a=b=1$

$m\geq 2$: $\chi(M^{st}_{d,d})=0 \quad \forall d\geq 2$

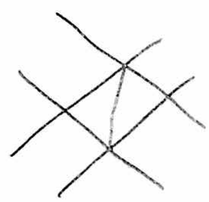
$$M^{st}_{1,1} \cong \mathbb{P}^{m-1} \Rightarrow \chi(M^{st}_{1,1}) = m-1$$

So

$$F(t) = 1 - t\,F(t)^{(m-1)^2}$$

<u>Claim</u>:

- This Functional Equation Comes From a Spectral Network Formed by gluing m-Copies of



together

- Functional equation For $d(3n, 2n, m)$ is degree 39 in F (w/ $\mathbb{Z}$-Coeffs)

DT invariants in Physics: $N=2$, $d=4$ Field Theory on $\mathbb{R}^{3,1}$

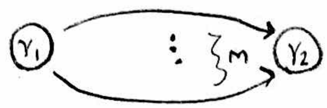
- $\exists$ a whole moduli Space (of Vacua) $\supset \mathcal{B}$
- $\hat{\Gamma} \longrightarrow \mathcal{B}$ local System of "Charge lattices" $\langle\cdot,\cdot\rangle$ $\quad \Gamma_u =$ e & m charges For theory over u
- BPS States: States "Stable" under deformations on $\mathcal{B}$.
- $\Omega(\gamma;u) =$ Weighted Count of BPS States of Charge $\gamma \in \Gamma_u$.

- Sometimes the spectrum is controlled by a "BPS quiver":

nodes $\longleftrightarrow$ + basis of Q

arrows $\longleftrightarrow$ $\langle \cdot , \cdot \rangle$ on Γ

e.g. $\Gamma \cong \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$ with $\langle \gamma_2, \gamma_1 \rangle = m$:



<u>Claim</u>:

$$\Omega(a\gamma_1 + b\gamma_2) = d(a,b,m) \qquad (a,b \in \mathbb{Z}^2).$$

(Pure SU(3)-SYM has Kronecker m-quiver as a sub-quiver).

Another technique to compute Ω's: <u>Spectral Nets</u>.

<u>Data</u>:

marked pts + "Asymptotic Data"

- Curve C w/ defects $\mathbb{D}$ $\rightsquigarrow$ 4D Theory $S[A_{K-1}, C, \mathbb{D}]$ (= T_K) on $\mathbb{R}^{3,1}$
- $K \geq 2$

w/ Coulomb branch $\mathcal{B}$

Compactify T_K:

Theory T_K^c on $\mathbb{R}^{2,1} \times S^1$ $\rightsquigarrow$ Moduli Space $(T_K^c) = \mathcal{M}^{SL_n}_{Hitchin}(C, \mathbb{D})$

(Point: $\cdot$ (E, ∇_E) hermitian v.b. w/ conn.
$\cdot$ $\varphi \in \Omega^1(C; \mathrm{End}(E))$)

$\mathbb{CP}^1$-worth of $\mathbb{C}$ str. on $\mathcal{M}_H$

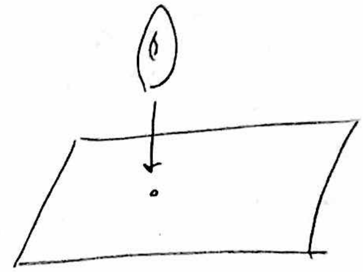
$\zeta = 0, \infty$: Higgs Bundles (∇_E, φ), $(\nabla_E, \bar{\varphi})$

$\zeta \neq 0, \infty$: Flat SL_K Connections $\nabla^{(\zeta)} = \nabla_E + \frac{1}{\zeta}\varphi + \zeta\bar{\varphi}$

$\mathcal{M}_{Higgs} \longrightarrow \mathcal{B}$ ← Coulomb branch

$\varphi \longmapsto \det(\eta - \varphi)$

$\uparrow$
K_C



So

$(\varphi, A) \in \mathcal{M}_{Higgs} \longleftrightarrow (\Sigma, \mathcal{L})$, $\Gamma \leq H_1(\Sigma; \mathbb{Z})$

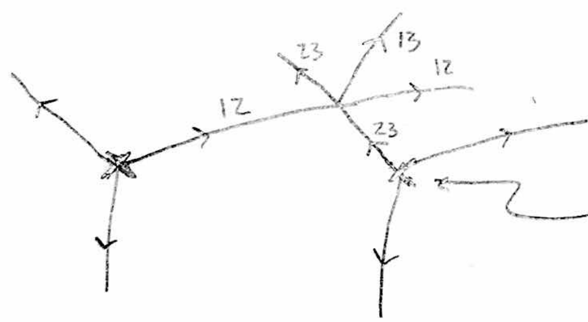
$\mathcal{L}$ ← Line bundle on Σ (eigenvectors)

$\Sigma \downarrow C$

$\Sigma = \{\eta : \det(\eta - \varphi) = 0\} \subseteq \text{Tot } K_C$

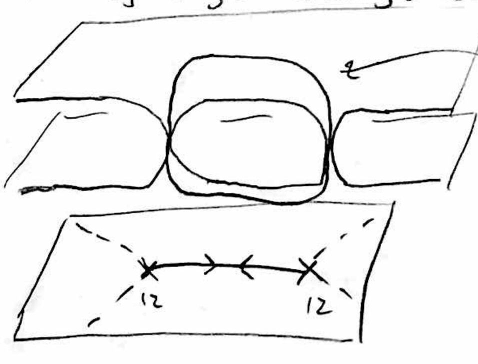
φ ⇝ System of Diff$^{\underline{l}}$ eqns on C whose integral curves are labeled by pairs of sheets of Σ ⊃ Spectral Network:

Ex:



← Branch points of Σ
$\downarrow$
C

Degenerate Network: ij edge and ji edge collide head on:



← lifts to a closed cycle on Σ of homology class $\gamma_c \in \Gamma$

$\Omega(n\gamma_c) =$ "# (Primitive) loops w/ hom. class $n\gamma_c$"

(Algebraic)

Spectral Network Machinery $\xrightarrow[\text{examples}]{\text{Known}}$ Functional Eqns For

$$F(t) = \prod_{n \geq 1} \left(1 - (-1)^{n a b} t^n\right)^{\Omega(n \gamma_c)}$$

$$\updownarrow$$

Algebraic over $\mathbb{Q}(t)$.