Entanglement is a Global Conspiracy Encoded in a Graded Vector Space

I. Punchline: $\exists$ an assignment

$$\begin{array}{ccccc} \text{multipartite (density) State} & \rightsquigarrow & \text{Chain Complex} & \xrightarrow{\text{homology}} & \text{Graded Vector Space} \\ \rho & & EC(\rho) & & EH(\rho) \end{array}$$

Such that $EH(\rho)$ detects how entangled ρ is.

II. Motivation:

1) Categorification of Mutual Information

(Decategorification: the process of taking a mathematically rich object and making a simpler object out of it at the cost of losing information)

Info. loss

$$\text{Rich Mathematical Object} \xrightarrow[\text{Decategorification}]{} \text{Simpler Mathematical Object}$$

Ex:

- Vector Space $\longrightarrow$ Dimension
- SUSY QM Hilbert Space $\longrightarrow$ Witten Index $\left(=\chi\left(0 \longrightarrow \mathcal{H}_B \xrightarrow{Q} \mathcal{H}_F \longrightarrow 0\right)\right)$
- Topological Space / Homology $\longrightarrow$ Euler Characteristic
- Khovanov homology of a Knot $\longrightarrow$ Jones polynomial

In these examples:

Categorification

$$\text{Graded V.S. } V = \bigoplus_{i=1}^{n} V_i \xrightarrow[\text{Characteristic}]{\text{Euler}} \chi(V) = \sum_{i=1}^{n} (-1)^i \dim(V_i)$$

$\updownarrow$

Homology Theory

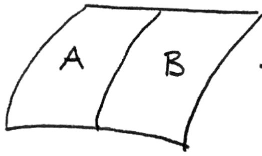
$\updownarrow$

(Chain Complex)

Q: Is there a Categorification of Von-Neumann Entropy / Mutual Information?

Mutual Information

Setup: Two Subsystems (bipartite System)



$\rightsquigarrow$ Hilbert Spaces

$\mathcal{H}_A , \mathcal{H}_B$

$\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$

(e.g. lattice sites; disjoint Causal diamonds, ...)

Fix a state $\rho_{AB} \in \mathrm{Dens}(\mathcal{H}_{AB}) \leq \mathrm{End}(\mathcal{H}_{AB})$

$$\rightsquigarrow \quad \rho_A := \mathrm{Tr}_{\mathcal{H}_B} \, \rho_{AB}$$

$$\rho_B := \mathrm{Tr}_{\mathcal{H}_A} \, \rho_{AB}$$

Then we define

$$S(\lambda) = \mathrm{Tr}_{\mathcal{H}_\lambda} [\rho_\lambda \log \rho_\lambda] \quad ; \quad \lambda \in \{A, B, AB\}$$

and the mutual info:

$$I(AB) = S(A) + S(B) - S(AB)$$

Generalization: N Subsystems $\{A_1, \dots, A_n\} = \mathcal{A}$

$$I(\mathcal{A}) := \sum_{\lambda \subseteq \mathcal{A}} (-1)^{n-|\lambda|-1} S(\lambda)$$

Psychometrika
Vol. 19, No. 2
W. McGill 1954

Looks like an Euler Characteristic!

$$(-1)^{n+1} I(\mathcal{A}) = \sum_{i=0}^{n} (-1)^{|\lambda|} \underbrace{\left[\sum_{|\lambda|=i} S(\lambda) \right]}_{\text{"dimension" of some "vector space"}}$$

Rmk: "$I(\Lambda)$ detects information that is Shared Completely among all Subsystems"
($I(\Lambda)$ detects the presence of independent Subsystems)

Claim: IF $\rho_\Lambda = \rho_\xi \otimes \rho_\eta$ For some $\xi, \eta \subseteq \Lambda$, then $I(\Lambda) = 0$.

ξ, η are "uncorrelated"

2)

$$\text{Entangled State} \longleftrightarrow \overset{(1)}{\text{Obstruction to Factorization}} \longleftrightarrow \overset{(2)}{\text{Global Correlations}}$$

(Correlations among disjoint subsystems)

Homology detects global obstructions

(1) $\longleftrightarrow$ (2): $\rho = \rho_A \otimes \rho_B \iff 0 = \mathrm{Cov}(\mathcal{O}_A, \mathcal{O}_B)$

$$= \mathrm{Tr}[\rho_{AB}\, \mathcal{O}_A \otimes \mathcal{O}_B] - \mathrm{Tr}[\rho_A \mathcal{O}_A]\,\mathrm{Tr}[\rho_B \mathcal{O}_B]$$

$\iff$ all information can be locally obtained

expectation values and higher moments of random variables / observables

---

(Co)-homology: Detects Global Obstructions

Key: Reformulate obstructions into an obstruction problem in linear algebra:

Suppose

$$V \xrightarrow{f} W \xrightarrow{g} X \quad \text{w/ } g \circ f = 0$$

Then

$$\mathrm{Ker}(g) / \mathrm{im}(f) = \text{Obstructions to Solving } v = f \cdot u \text{ given: } g \cdot v = 0.$$

Def

1)
$$C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n$$
is a Chain Cx if $d_i d_{i-1} = 0$.

2) $H^i := \mathrm{Ker}(d_i) / \mathrm{Im}(d_{i-1})$

Ex: • $C^i = \Omega^i(M)$ = i-Forms on a manifold M

H^i = de-Rham Cohomology

Suppose $d\omega = 0$ for $\omega \in \Omega^1$

Then is $\omega = df = \frac{\partial f_i}{\partial x_i} dx_i$?

Yes in $\mathbb{R}$: $f(t) = \int_{t_0}^{t} \omega$; not nec on S^1 : $H^1(S^1) \neq 0$.

• Linearized Cohomology of S^1 : (Refer to Jan. 2015 notes)

Entanglement (Co)-homology

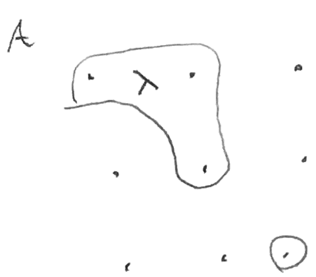
Setup: • N "primitive" Subsystems $\{A_1, \ldots, A_N\} = A$

• State $\rho_A \in \mathrm{Dens}(\mathcal{H}_A)$ $(\mathcal{H}_A := \mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_N})$

To each subsystem $\lambda \subseteq A$ we have an assignment

$$\mathrm{Var} : \lambda \longmapsto \mathrm{Im}(\rho_\lambda) \otimes \mathrm{Im}(\rho_\lambda)^*$$
$$\phi \longmapsto \mathbb{C}$$

"Space of random-Vars / Observables detectable by ρ_λ"



Then take

$$\begin{array}{c} C^{\ell} := \bigoplus\limits_{\substack{\lambda \leq \Lambda \\ |\lambda| = \ell+1}} \mathrm{Var}(\lambda) \\ \Big\downarrow \partial \\ C^{\ell+1} \end{array}$$

For every $|\lambda| = \ell+1$ an assignment of an observable

What is ∂? First note:

$$\begin{tikzcd} & \mathrm{End}(\mathcal{H}_\lambda) \arrow[r, "\otimes \mathbb{1}_{\eta\setminus\lambda}"] & \mathrm{End}(\mathcal{H}_\eta) \arrow[d, "\mathrm{proj}"] \\ (\lambda \hookrightarrow \eta) \arrow[r, mapsto] & \mathrm{Var}(\lambda) \arrow[u, "\text{include}"'] \arrow[r, dashed, "\mathrm{Var}(\lambda \hookrightarrow \eta)"] & \mathrm{Var}(\eta) \end{tikzcd} \quad = \text{“} \otimes \mathbb{1}_{\eta\setminus\lambda} \text{”}$$

(Claim: Functorality $\mathrm{Var}(\lambda \hookrightarrow \eta \hookrightarrow \xi) = \mathrm{Var}(\eta \hookrightarrow \xi) \circ \mathrm{Var}(\lambda \hookrightarrow \eta)$)

Example For bipartite System

$$\begin{array}{ccc} 0 & & \\ \Big\downarrow & & \\ \mathbb{C} \cong C^{-1} & & a \\ \partial \Big\downarrow & & \Big\downarrow \\ C^{0} & & (a\mathbb{1}_A, a\mathbb{1}_B) \\ \partial \Big\downarrow & & \\ C^{1} & & \\ \partial \Big\downarrow & & \\ 0 & & \end{array}$$

$S \in C^0 \implies S: \begin{array}{l} A \longmapsto \mathcal{O}_A \\ B \longmapsto \mathcal{O}_B \end{array}$

$S \in C^1 \implies S: AB \longmapsto \mathcal{O}_{AB}$

So if $S \in C^0$:

$$\partial S : AB \longmapsto \text{“}\mathbb{1}_A \otimes \mathcal{O}_B\text{”} - \text{“}\mathcal{O}_A \otimes \mathbb{1}_B\text{”} \in C^1 \cong \mathrm{Im}(\rho_{AB}) \otimes \mathrm{Im}(\rho_{AB})^*$$

Note:
- $\partial S = 0 \implies \mathcal{O}_A$ and $\mathcal{O}_B$ give the same outcome when measured

$$\Updownarrow$$

$$\mathbb{1} \otimes \mathcal{O}_B \sim \mathcal{O}_A \otimes \mathbb{1}$$

$$\Updownarrow$$

$$\mathrm{Tr}[\rho_{AB} (\mathbb{1} \otimes \mathcal{O}_B) \cdot C] = \mathrm{Tr}[\rho_{AB} (\mathcal{O}_A \otimes \mathbb{1}) \cdot C]$$

$$\forall C \in \mathrm{End}[\mathrm{Im}(\rho_{AB})] \quad (\text{or } \mathrm{End}(\mathcal{H}_{AB}))$$

- $S = \partial t \implies S = (a\mathbb{1}_A, a\mathbb{1}_B)$

$\Rightarrow \quad H^{-1} = \mathbb{C} \cdot \mathbb{1}$ = Constant random Variables

H^0 = Pairs of non-Constant random Variables $(\mathcal{O}_A, \mathcal{O}_B)$ s.t. $\mathcal{O}_A$ and $\mathcal{O}_B$ are Correlated

In general: For multipartite Systems (N-partite)

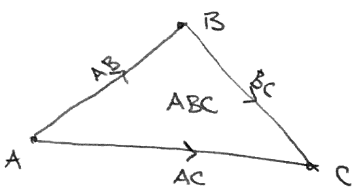
$H^K = \binom{N}{K+1}$ tuples of $(K+1)$-body operators / random Variables

$\longleftrightarrow \quad S: \lambda \longrightarrow \mathcal{O}_\lambda \;,\; |\lambda| = K+1$

Which are Correlated.

(the Space of $(K+1)$-body operators that are Completely Correlated among all $(K+1)$-Subsystems in the presence of ρ.)

Ex: 3-partite System



$\rho \in C^1$

$$\partial\rho(ABC) = \mathcal{O}_{BC} - \mathcal{O}_{AC} + \mathcal{O}_{AB}$$

$$\partial\rho = 0 \Rightarrow \mathcal{O}_{AC} \sim \mathcal{O}_{BC} + \mathcal{O}_{AB}$$

$\rho = \partial\alpha \Rightarrow \rho$ can be reduced to 1-observer Variables

---

Ex: bipartite qubit System with

$$\rho_{AB} = |\psi\rangle\langle\psi| \;,\; |\psi\rangle = |1_A 0_B\rangle + |0_A 1_B\rangle$$

Then

$$\left.\begin{array}{l} (|1_A\rangle\langle 1_A|, |0_B\rangle\langle 0_B|) \\ (|0_A\rangle\langle 0_A|, |1_B\rangle\langle 1_B|) \end{array}\right\} \in C^1$$

have non-trivial image in H^0,

$$\dim H^0 = 6.$$

ρ pure.

<u>Conjecture</u>: $\check{H}^i_\rho = 0 \iff \rho_{\mathcal{A}} = \rho_{A_K} \otimes \rho_\eta$ for some $A_K \in \mathcal{A}$

<u>Corollary</u>: $\rho = \rho_{A_K} \otimes \rho_\eta \implies$
$$\chi_\rho = \sum_{i=0}^{n} (-1)^i \dim H^i_\rho = 0$$

<u>Remark</u>

- χ_ρ can also be calculated from the chain complex:

$$\chi_\rho = \sum_{i=0}^{n} (-1)^i \dim C^i_\rho$$
$$= \sum_{\lambda \subseteq \mathcal{A}} (-1)^{(|\lambda|+1)} \operatorname{rank}(\rho_\lambda)^2 \qquad (*)$$

One can easily prove the Corollary from $(*)$.

- $\chi_\rho = 0 \not\Rightarrow \rho = \rho_{A_K} \otimes \rho_\eta$, e.g. $\rho = |\psi\rangle\langle\psi|$ for
$$|\psi\rangle = |0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle$$

(Can see $\rho = 0$ from $(*)$) nevertheless $H_\rho \cong \mathbb{C}^7[0] \oplus \mathbb{C}^7[1] \neq 0$.

<u>Extra Remarks</u>

- There is a Künneth theorem:

$$C^\bullet_{\rho_{A_1} \otimes \rho_{A_2}} = C^\bullet_{\rho_{A_1}} \otimes C^\bullet_{\rho_{A_2}}$$
$$\implies H^\bullet_{\rho_{A_1} \otimes \rho_{A_2}} = H^\bullet_{\rho_{A_1}} \otimes H^\bullet_{\rho_{A_2}}$$

allows us to prove $H^\bullet_\rho = 0$ if $\rho = \rho_{A_K} \otimes \rho_\eta$.