Talk III: Homological Toolkit for the Quantum Mechanic

Recap

Talk I: • Is multipartite mutual info an Euler characteristic?

• We defined the State Index

– $\underline{\rho_P} = \left((\mathcal{H}_S)_{S \in P},\ \rho \in \mathrm{Dens}\left(\bigotimes_{S \in P} \mathcal{H}_S \right) \right)$

– $\chi(\underline{\rho_P}) : \mathbb{C}^3 \longrightarrow \mathbb{C}$

$$(\alpha, q, r) \longmapsto \sum_{\emptyset \subsetneq T \subseteq P} (-1)^{|T|} \dim(\mathcal{H}_T)^{\alpha} \left(\mathrm{Tr}[\rho_T^{\,q}] \right)^r$$

$$\begin{array}{ccc} & \chi(\underline{\rho_P}) & \\ \swarrow{\scriptstyle \lim_{q \to 1} \frac{\chi}{(q-1)^r}} & & \searrow{\scriptstyle q \to 0} \\ I(\underline{\rho_P}) = \sum_{T \subseteq P} (-1)^{|T|-1} S(\rho_T) & & \sum_{T \subseteq P} (-1)^{T} \dim(\mathcal{H}_T)^{\alpha} \, \mathrm{rank}(\rho_T)^r \end{array}$$

Properties:

1) $\chi(\underline{\rho_P}) \in \mathcal{O}(\mathbb{C}^3)$

2) $\chi(\underline{\rho_P} \otimes \underline{\varphi_Q}) = \chi(\underline{\rho_P})\, \chi(\underline{\varphi_Q})$

3) $\underbrace{\chi(\underline{\rho_P})}_{|P|\text{-partite}} = -\Big[\underbrace{\chi(\rho_{\partial_i P}) + \chi(\rho_{\partial_j P}) - \chi[\lambda_{ij}[\underline{\rho_P}]]}_{|P|-1 \text{ partite}} \Big]$

Intuition: – χ acts like the Euler characteristic of a complex $\underline{\rho_P}$

$$\underline{\rho_P} = \quad 0 \longrightarrow \mathbb{C} \longrightarrow \underset{|T|=1}{\boxplus} \underline{\rho_T} \longrightarrow \underset{|T|=2}{\boxplus} \underline{\rho_T} \longrightarrow \cdots \longrightarrow \underset{|T|=|P|}{\boxplus} \underline{\rho_T} \longrightarrow 0$$

$$\wr$$

$$\dim = \sum_{|T|=1} \dim(\mathcal{H}_T)^{\alpha} \left[\mathrm{Tr}(\rho_T^{\,q}) \right]^r$$

Q: Can we make this Statement precise?

(Need a Category of "multipartite States")

Easier: Take $q \to 0$, then $\dim = \dim(\mathcal{H})^{\alpha} \operatorname{rank}(p)^{r} \in \mathbb{Z}$

$(\alpha, r \in \mathbb{Z})$

$\Rightarrow$ possible Complex of Vector Spaces.

Talk II:

- Defined the GNS module: $\hat{p} \in \mathrm{Dens}(\mathcal{H})$

$$\mathrm{GNS}(\hat{p}) \cong \underset{\underset{\mathrm{End}(\mathcal{H})}{\circlearrowright}}{\mathcal{H}} \otimes \mathrm{Image}(p)^{\vee}$$

$\ni S_{\hat{p}}$: Support proj. of $\hat{p}$

$\dim = \dim(\mathcal{H}) \operatorname{rank}(p)$

$\cong$ "Right Essential Equivalence Classes" of ops

- Thm: $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is Factorizable $\Leftrightarrow$ $H^{\circ}(G_{\psi}) = 0$

where

$$G_{\psi} = \mathbb{C} \xrightarrow{d^{-1}} \mathrm{GNS}(p_A) \times \mathrm{GNS}(p_B) \xrightarrow{d^{0}} (\mathcal{H}_A \otimes \mathcal{H}_B) \otimes (\mathrm{Span}_{\mathbb{C}} \psi)^{\vee} \cong \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\lambda \longmapsto (\lambda S_{\hat{p}_A}, \lambda S_{\hat{p}_B})$$

$$(a, b) \longmapsto (a \otimes I_B - I_A \otimes b) S_{\psi \otimes \psi^{\vee}}$$

This Talk: Multipartite Complexes

Setup:

- A a W^*-alg, e.g. $A = \prod_{i=1}^{N} M_{n_i} \mathbb{C}$
- $p: A \to \mathbb{C}$ a positive linear $F^{\underline{n}\underline{l}}$, e.g.

$$p(a_1, \ldots, a_N) = \sum_{i=1}^{N} \mathrm{Tr}[\hat{p}_i a_i]$$

- S_p the Support proj. of p:

$$S_p = (S_{\hat{p}_1}, \ldots, S_{\hat{p}_N}) \in A$$

Then the GNS module is defined by

$$\begin{aligned} GNS(\rho) &= A/\mathcal{I}_\rho , \quad \mathcal{I}_\rho = \{a \in A : \rho(a^* a) = 0\} \\ &= A/\sim , \quad a \sim b \text{ if } \rho(x^* a) = \rho(x^* b) \; \forall x \in A \\ &\cong A S_\rho \\ &\Big(\underset{\substack{\text{Finite} \\ \text{dim'l density}}}{\cong} \mathcal{H} \otimes \mathrm{Image}(\rho)^\vee \Big) \end{aligned}$$

Now, given a multipartite state over a set of $\otimes$-factors P

$$\underline{\rho_P} = \Big((A_s)_{s \in P} , \; \rho_P : \bigotimes_{s \in P} A_s \longrightarrow \mathbb{C} \Big)$$

Our goal is to describe the process

$$\underline{\rho_P} \overset{(A)}{\rightsquigarrow} \underset{\text{Functor}}{\mathcal{G} : \mathrm{Subsets}(P) \longrightarrow \mathrm{Vect}} \underset{\text{Čech}}{\overset{(B)}{\rightsquigarrow}} \mathrm{Cohomology}$$

GNS - modules on reduced states

$$T \longmapsto GNS(\rho_T)$$

$$T \hookrightarrow V \longmapsto GNS(\rho_T) \longrightarrow GNS(\rho_V)$$

$$\left(\text{s.t. } \mathcal{G}(T \hookrightarrow V \hookrightarrow W) = \mathcal{G}(V \hookrightarrow W) \circ \mathcal{G}(T \hookrightarrow V) \right)$$

Process (A):

reduced state on $T \subseteq P$

- What is ρ_T ?

$$\iota_T : \underbrace{\bigotimes_{t \in T} A_t}_{A_T} \longrightarrow \underbrace{\bigotimes_{s \in P} A_s}_{A_P}$$

$$a \longmapsto a \otimes 1_{T^c}$$

$$\rho_T := \iota_T^* \rho_P : A_T \longrightarrow \mathbb{C}$$

$$a \longmapsto \rho(a \otimes 1_{T^c})$$

Define

$$\mathcal{G}(T) := GNS(\rho_T)$$

Want

$\mathbb{C}$-linear map

$$G : (T \hookrightarrow V) \longmapsto G(T) \longrightarrow G(V)$$

Observation: Let $\rho, \varphi : A \longrightarrow \mathbb{C}$ (positive linear functionals)

s.t. $S_\rho \leq S_\varphi$ $(S_\varphi S_\rho = S_\rho)$

then $\exists$ a morphism

$$GNS(\varphi) \cong A S_\varphi \longrightarrow A S_\rho \cong GNS(\rho)$$

$$a S_\varphi \longmapsto (a S_\varphi) S_\rho = a S_\rho$$

(Equivalently $\mathcal{I}_\varphi \leq \mathcal{I}_\rho \Rightarrow A/\mathcal{I}_\varphi \longrightarrow A/\mathcal{I}_\rho$)

Lemma: (Compatibility of Supports)

Suppose $\rho_{AB} : A \otimes B \longrightarrow \mathbb{C}$

then $S_{\rho_{AB}} \leq S_{\rho_A \otimes \rho_B} \quad (= S_{\rho_A} \otimes S_{\rho_B})$

PF: $\mathcal{I}_\rho = A(1 - S_\rho)$ So the lemma is equiv. to the Statement.

$$\mathcal{I}_{\rho_{AB}} \leq \mathcal{I}_{\rho_A \otimes \rho_B} = \overline{\mathcal{I}_{\rho_A} \otimes \mathcal{I}_{\rho_B}}$$

Take $z \in \mathcal{I}_{\rho_A}$, then

$$0 = \rho_A(z^* z) = \rho_{AB}(z^* z \otimes \mathbb{1})$$
$$= \rho_{AB}\left[(z \otimes \mathbb{1})^*(z \otimes \mathbb{1})\right]$$

$\Rightarrow z \otimes \mathbb{1} \in \mathcal{I}_{\rho_{AB}}$

- Use the fact that $\mathcal{I}_{\rho_{AB}}$ is an ideal to show $\mathcal{I}_{\rho_A} \otimes \mathcal{I}_{\rho_B} \leq \mathcal{I}_{\rho_{AB}}$.

$\square$

Thus, we have maps

$$\begin{array}{ccccc} GNS(\rho_A) & \longrightarrow & GNS(\rho_A \otimes \rho_B) & \longrightarrow & GNS(\rho_{AB}) \\ a & \longmapsto & a \otimes S_B & \longmapsto & (a \otimes S_B) S_{AB} \\ & & (a \otimes \mathbb{1} \bmod \mathcal{I}_{\rho_A \otimes \rho_B}) & & \end{array}$$

So we define

$$\begin{tikzcd} & & GNS(\rho_T \otimes \rho_{T \setminus V}) \arrow[d] \\ G(T \hookrightarrow V) = & GNS(\rho_T) \arrow[ur, dashed] \arrow[r] & GNS(\rho_V) \\ & a \arrow[r, mapsto] & (a \otimes S_{V \setminus T}) S_V \end{tikzcd}$$

Claim Compatibility of Supports

$$\Rightarrow \quad G(T\hookrightarrow V\hookrightarrow W) = \underbrace{G(V\hookrightarrow W)\circ G(T\hookrightarrow V)}$$

$$\begin{array}{llcl} \Big(a_A \longmapsto & (a\otimes S_{BC})S_{ABC} & = & [(a\otimes S_B)S_{AB}\otimes S_C]S_{ABC} \\ & \shortparallel & & \shortparallel \\ & (a\otimes S_B\otimes S_C)(S_A\otimes S_{BC})S_{ABC} & = & (a\otimes S_B\otimes S_C)S_{ABC} \end{array}$$

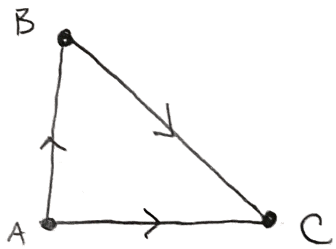
$\Rightarrow$ G is a Functor $\quad (G(\emptyset) = G(p_\emptyset) = G(1) \cong \mathbb{C})$

(B): From Functors $\mathcal{F}: \mathrm{Subsys}(P) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$

to Cohomology:

$$\mathcal{F} \rightsquigarrow \mathcal{F}(\emptyset) \xrightarrow{d^{-1}} \overset{C^0(\mathcal{F})}{\overset{\shortparallel}{\prod_{\substack{T\subseteq P \\ |T|=1}} \mathcal{F}(T)}} \xrightarrow{d^0} \overset{C^1(\mathcal{F})}{\overset{\shortparallel}{\prod_{|T|=2} \mathcal{F}(T)}} \xrightarrow{d^1} \cdots \xrightarrow{d^{|P|-2}} \prod_{|T|=|P|-1} \mathcal{F}(T) \xrightarrow{d^{|P|-1}} 0$$

Picture: Draw a $(|P|-1)$-Simplex whose vertices are labelled by the ordered set P:



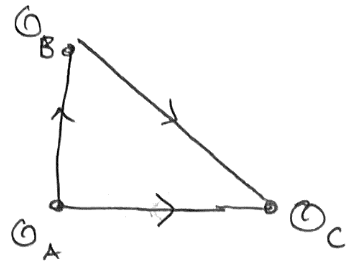
$(P = \{A < B < C\})$

$\mathcal{F} \rightsquigarrow$ Assignment of Vector Spaces to Faces:

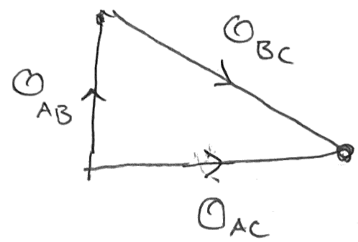
$$F_A \longmapsto \mathcal{F}(A)$$
$$F_{AB} \longrightarrow \mathcal{F}(AB)$$
$$\vdots$$

$C^k(\mathcal{F})$ = Section of assignment over k-skeleton

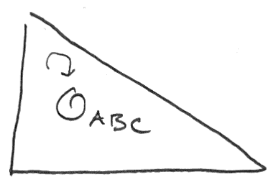
$\mathcal{O} \in C^0(\mathcal{F})$:



$\mathcal{O} \in C^1(\mathcal{F})$



$\mathcal{O} \in C^2(\mathcal{F})$

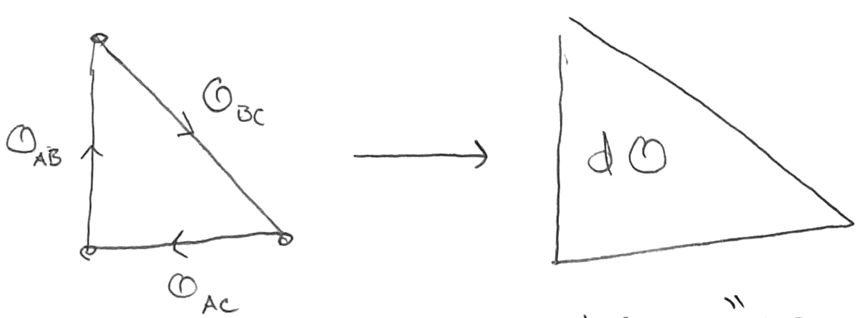


$$d^k : C^k(\mathcal{F}) \longrightarrow C^{k+1}(\mathcal{F})$$

$$\mathcal{O} \longmapsto d^k\mathcal{O} : \underbrace{V}_{(k+1)\text{-Face}} \longmapsto \sum_{j=0}^{k+1} (-1)^j \underbrace{\left(\quad\right)}_{\mathcal{F}(\partial_j V \to V)} \mathcal{O}_{\partial_j V}$$

$\partial_j V$: j^{th} ∂ of V

Ex:



$d\mathcal{O} =$ " $\mathcal{O}_{AB} + \mathcal{O}_{BC} - \mathcal{O}_{AC}$ "

suppressing embeddings

- Apply this to $G_{\underline{\rho_P}} : \mathrm{Subsys}(P) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$

$$\rightsquigarrow C^\bullet[G_{\underline{\rho_P}}]$$

<u>Thm</u>:

$$C^\bullet[G_{\underline{\rho_P} \otimes \underline{\varphi_Q}}] \cong \left[C^\bullet[G_{\underline{\rho_P}}] \otimes C^\bullet[G_{\underline{\varphi_Q}}]\right][1]$$

PF:

$$\mathrm{GNS}(\rho \otimes \varphi) \cong \mathrm{GNS}(\rho) \otimes \mathrm{GNS}(\varphi)$$

<u>Cor</u>: Same iso in Cohomology (Künneth Thm For Chain Complexes Over Char 0 Fields)

<u>Cor^2</u>: $\underline{\rho_P} = \bigotimes_{S \in P} \underline{\rho_S}$ ← Unipartite States

$$\Rightarrow \quad HG^\bullet(\underline{\rho_P}) \cong \left(\bigotimes_{S \in P} H^0(\underline{\rho_S})\right)[|P|-1]$$

$$\cong \left[\mathbb{C}^{\mathrm{rank}(\rho_S)\dim(\mathcal{H}_S)-1}\right][|P|-1]$$

$\Rightarrow$ Cohomology Concentrated in top degree.

<u>Cohomology and Correlation</u>

<u>Q</u>: What is H^k telling us?

Look @ a $(k+1)$-Face F_V, $V \subseteq P$ and look at Sections $\mathcal{O} \in \Gamma_G(\partial F_V)$ S.t.

$$(d^k \mathcal{O}|_V) = \sum_{l=0}^{k+1} (-1)^l \mathcal{O}_{\partial_k V} \sim_{\rho_V} \mathcal{O} \qquad (*)_{\underline{\rho_P}}$$

$\Rightarrow$ Possible non-local Correlations

E.g. $V = AB$

$$\mathcal{O}_B - \mathcal{O}_A \sim_{\rho_{AB}} 0$$

But there are "dumb" solutions to $(*)$:

- Ex: $\mathbb{1}_A \otimes \mathbb{1}_B - \mathbb{1}_A \otimes \mathbb{1}_B \sim 0$

Trivial Solutions := Solutions to $(*)$ wrt to Fully Factorizable State

$$\bigotimes_{s \in P} \underline{\rho_s}$$

$$\mathrm{Cor}_V(\underline{\rho_P}) = \text{Solutions to } (*) \text{ for } \underline{\rho_P} \Big/ \text{Trivial Solutions}$$

<u>Def</u>: $\mathcal{O} \in C^k(G_P)$ exhibits non-trivial correlations along V $(|V| = k+1)$

if $\underbrace{[\mathcal{O}|_{\partial V}]}_{\in \mathrm{Cor}_V(\underline{\rho_P})} \neq 0$

<u>Claim</u>

$$H^k[G_P] = \overbrace{\left\{ \begin{array}{c} \text{Sections over } k\text{-Skeleton} \\ \text{that exhibit possible} \\ \text{Correlations along each} \\ \text{Face} \end{array} \right\}}^{\text{Cocycles}} \Bigg/ \overbrace{\left\{ \begin{array}{c} \text{Sections that} \\ \text{exhibit trivial} \\ \text{Correlations along} \\ \text{each Face} \end{array} \right\}}^{\text{Coboundaries}}$$

Solutions to $(*)$

trivial Solutions to $(*)$

$\mathcal{O} \in C^k(G_P)$ has $[\mathcal{O}] \in H^k$ non-zero

$$\Updownarrow$$

$$\exists \underset{|V| = k+1}{V \subseteq P} \text{ s.t. } \underbrace{[\mathcal{O}|_{\partial V}]}_{\in \mathrm{Cor}_V(\underline{\rho_P})} \neq 0.$$