Rutgers Subgroup Meeting 10/13/2017 (Friday 13th)

Big Machinery in QM: An intro to Cohomology as a tool for quantifying multipartite entanglement

I. Conclusions

- Multipartite entanglement seems to be captured with Cohomological algebra. Roughly:

$$\text{"} H^{k}\left(\begin{array}{c}\text{n-partite density}\\ \text{State}\end{array}\right) = \Big\{ \underbrace{\left(\mathcal{O}_1, \dots, \mathcal{O}_{\binom{n}{k}}\right)}_{\text{Correlated k-body ops.}} \Big\} \text{"}$$

"Finest partition"

($H^{\cdot} \equiv 0$ if state is factorizable)

- $$H^{\cdot} \xrightarrow[\text{Lefschetz-Index}]{\text{Euler Characteristic}} \text{q-deformation of Multi-Variate Mutual Info} \in \mathbb{R}$$

$q \to 1$: Multivariate Mutual Inf. $\in \mathbb{R}$

$q \to 0$ (actually $\left.\frac{d}{dq}\right|_{q\to 0}$): $\mathbb{Z}$-Valued quantity

II. Multivariate Factorization

Setup:

- $(\mathcal{H}_i)_{i\in I}$ Collection of Hilbert spaces; I a finite ordered set $|I| = n$.
- $\hat{\rho}_I \in \mathrm{Dens}\left(\bigotimes_{i\in I} \mathcal{H}_i\right)$ ← Trace = 1 endomorphisms

Def:

- $\hat{\rho}_T = \mathrm{Tr}_{I\setminus T}(\hat{\rho}_I)$, $T \subseteq I$
- $\mathcal{E}_T := \mathrm{End}^b\left(\bigotimes_{t\in T} \mathcal{H}_t\right)$
- $\mathbb{E}_T(\cdot) = \mathrm{Tr}\left[\hat{\rho}_T(-)\right]$

Def: • Let $\lambda = (\overset{\subseteq I}{\lambda_1}, \dots, \overset{\subseteq I}{\lambda_\ell})$ be a partition of I, then

$\hat{\rho}_I$ is λ-Factorizable

$$\Updownarrow$$

$$\hat{\rho}_I \text{ “}=\text{” } \hat{\rho}_{\lambda_1} \otimes \cdots \otimes \hat{\rho}_{\lambda_\ell} \text{”} \longleftrightarrow \mathbb{E}_I(a_{\lambda_1} \otimes \cdots \otimes a_{\lambda_n}) = E_{\lambda_1}(a_{\lambda_1}) \cdots E_{\lambda_n}(a_{\lambda_n})$$

$$\forall a_{\lambda_k} \in \mathcal{E}_{\lambda_k}.$$

• (Weakly) λ-entangled if <u>not</u> λ-Factorizable.

(Rmk: λ-Factorizable $\Leftrightarrow$ λ' Factorizable $\forall$ coarsenings λ'

$$\updownarrow$$

λ-entangled $\Leftrightarrow$ λ' entangled $\forall$ refinements λ')

III. Why Cohomology?

Factorizability $\longleftarrow\!-\!-\!\rightarrow$ Descent of data to Subsystems : all global data comes from "gluing" local data

$$\updownarrow$$

$\mathbb{E}(\omega) = \int_M \omega$ constant under perturbations

$\uparrow$

$\omega^d, (d\omega^d) = 0$

Entanglement $\longleftarrow\!-\!-\!\rightarrow$ Obstruction to Descent ← Typically captured Cohomologically

$\Updownarrow$

$\mathbb{E}_{A\cup B}(a_{\lambda_1} \otimes a_{\lambda_2}) \neq \mathbb{E}_A(a_{\lambda_1}) \mathbb{E}_B(a_{\lambda_2})$

E.g. de Rham Cohomology measures obstr. to $f(t_1) - f(t_0) = \int_{t_0}^{t_1} \omega'$

Exact Sequences For pure Factorizable States

Let $\psi = \psi_A \otimes \psi_B$, then we have maps

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\lambda \mapsto (\lambda\psi_A,\, \lambda\psi_B)"] & \mathcal{H}_B \times \mathcal{H}_A \arrow[r, shift left, "(\alpha,\beta) \mapsto \psi_1 \otimes \beta"] \arrow[r, shift right, "(\alpha,\beta) \mapsto \alpha \otimes \psi_2"'] & \mathcal{H}_A \otimes \mathcal{H}_B \end{tikzcd}$$

Similarly

$$\begin{tikzcd} & & & \prod_{i_1 < \cdots i_{n-1}} \mathcal{H}_{1 \cdots \hat{i}_1 \cdots \hat{i}_2 \cdots \hat{i}_{n-1} \cdots n} \arrow[d, phantom, "\parallel"] \\ \mathbb{C} \arrow[r, "d_{-1}"] & \mathcal{H}_n \times \cdots \times \mathcal{H}_1 \arrow[r, shift left] \arrow[r, shift right] & \cdots & \prod_{i<j} \mathcal{H}_{1 \cdots \hat{i} \cdots \hat{j} \cdots n} \arrow[d] \\ & & & \prod_{i=1}^{n} \mathcal{H}_{1 \cdots \hat{i} \cdots n} \arrow[d] \\ & & & \mathcal{H}_{1 \cdots n} \end{tikzcd}$$

$\psi = \psi_1 \otimes \cdots \otimes \psi_K \rightsquigarrow$

$\psi_i \in \mathcal{H}_i$

"labelled by Forgotten index"

n-1 arrows

n-arrows

Defined via

$$\begin{tikzcd} \prod_{i<j} \mathcal{H}_{1 \cdots \hat{i} \cdots \hat{j} \cdots n} \arrow[r, "Pr_{(\hat{i}_0 \hat{j}_0)}"] & \mathcal{H}_{1 \cdots \hat{i}_0 \cdots \hat{j}_0 \cdots n} \arrow[d, "\otimes \psi_{j_0}"] \\ & \mathcal{H}_{1 \cdots \hat{i}_0 \cdots n} \end{tikzcd}$$

can run over n-1 values

Claim: Take alternating sums of arrows, then

$$\mathbb{C} \xrightarrow{d_{-1}} \mathcal{H}_n \times \cdots \times \mathcal{H}_1 \xrightarrow{d_0} \cdots\cdots \xrightarrow{d_{n-2}} \mathcal{H}_{1 \cdots n}$$

Is exact: $\mathrm{Ker}(d_i) = \mathrm{Im}(d_{i-1})$

Bipartite Case: $\psi_1 \otimes \beta - \alpha \otimes \psi_2 = 0 \iff \alpha = \psi_1$, $\beta = \psi_2$

Thm

A State $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is Factorizable $\iff \exists$ (For $X = A, B$)

- left $\mathcal{E}_X$-modules M_X
- Distinguished points $m_X \in M_X$
- Equivariant maps $\Gamma_X : M_X \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_B$

S.t.

1. $\mathcal{E}_X m_X = M_X$ (cyclic)
2. $\Gamma_X(m_X) = \psi$
3. $$0 \longrightarrow \mathbb{C} \xrightarrow{\lambda \mapsto (\lambda m_A, \lambda m_B)} M_A \times M_B \xrightarrow{d_0 = \Gamma_A - \Gamma_B \,=\, pr_B^* \Gamma_A - pr_A^* \Gamma_B} \mathcal{H}_A \otimes \mathcal{H}_B$$

is exact. $(H^0 = 0)$.

Always a Complex

Thus given $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ we want to Find (M_X, m_X, Γ_X) w/ Smallest associated Cohomology.

Def: Let $\hat{\rho} \in \mathrm{Dens}(\mathcal{H})$

$$G_{\hat{\rho}} := \mathcal{E}_{\mathcal{H}} \big/ \{a : a \cdot \hat{\rho} = 0\} = \mathcal{E}_{\mathcal{H}} \big/ \{a : E_{\hat{\rho}}(a^* a) = 0\}$$

$$g_{\hat{\rho}} := [1_{\mathcal{H}}]$$

Left $\mathcal{E}_{\mathcal{H}}$-module + Cyclic Vector

Note:
- $G_{\hat{\rho}} \cong \mathrm{Hom}(\mathcal{H}, \mathrm{Image}(\hat{\rho})) \overset{\hat{\rho}\text{ pure}}{\rightsquigarrow} \cong \mathcal{H}$
- Can be equipped w/ inner product given by $E_{\hat{\rho}}$: GNS Rep^n

Thm: $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B \iff$ The Sequence

$$\begin{array}{ccccc} 1 & \longmapsto & ([I_A], [I_B]) & & \\ \mathbb{C} & \longrightarrow & G_A \times G_B & \xrightarrow{d_0} & G_{AB} \\ & & (a, b) & \xmapsto{d_0} & [a \otimes 1_B - 1_A \otimes b] \end{array}$$

Is exact $(H^0 = 0)$

Facts: • $H^0_{\hat{\rho}} = \{(a,b) \in G_A \times G_B : Cov(a,b) = Var_A(a) = Var_B(b)\}$

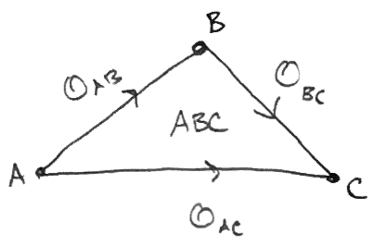
• $\hat{\rho} = \psi \otimes \psi^\vee \Rightarrow$ $\dim H^0 = r^2 - 1$

$\dim H^1 = (d_A - r)(d_B - r)$ $\quad d_X = \dim \mathcal{H}_X$

Where

$$r = \operatorname{rank}(\rho_X) = \text{Schmidt rank} : \psi = \sum_{i=1}^{r} \sqrt{\lambda_i}\, \alpha_i \otimes \beta_i$$

Tripartite: $\mathbb{C} \longrightarrow G_C \times G_B \times G_A \rightrightarrows G_{BC} \times G_{AC} \times G_{AB} \overset{+}{\underset{+}{\overset{-}{\rightrightarrows}}} G_{ABC}$



$O \in C^1$

$(dO)_{ABC} = O_{AB} + O_{BC} - O_{AC}$

$H^1 = \{(O_{AB}, O_{BC}, O_{AC}) : (dO)_{ABC} = 0\}$

IV: Categorification of Mutual Information

Categorification

Number

Chain Complex $(V_\bullet, d: V_\bullet \to V_{\bullet+1}) \xrightarrow{\text{Euler-char}} \chi(V) = \sum_{k \in \mathbb{Z}} (-1)^k \dim V_k$

$H^\bullet(-)$ (homology) $\longrightarrow$ Graded V.S $\longrightarrow \chi(V)$

More info $\qquad\qquad$ less info

Multivariate Mutual Info.

Data: • $\rho_I \in \mathrm{Dens}\left(\bigotimes_{i\in I} \mathcal{H}_i\right)$

• $\lambda_* =$ Finest partition

$$I^{\lambda_*}(\rho) = -\sum_{\emptyset \subseteq T \subseteq I} (-1)^{|I|-|T|} \underbrace{S_{vN}(\rho_T)}_{-\mathrm{Tr}[\rho_T \log \rho_T]} \in \mathbb{R}$$

Thm: ρ is λ-Factorizable $\iff I^{\lambda'}(\rho) = 0 \quad \forall\, \lambda' \geq \lambda$

(A bit overkill it is sufficient that $\sum_\ell S_{vN}(\rho_{\lambda_\ell}) - S(\rho_I) = 0$)

Note:

$$I^{\lambda_*}(\rho) = (-1)^{|I|+1} \sum_{k=0}^{|I|} (-1)^k \Bigg[\underbrace{\sum_{|T|=k} S_{vN}(\rho_T)}_{\text{looks like a ``dimension'' of } \rho_T} \Bigg]$$

Caveat: In order for the Euler-Characteristic to Factorize through Homology we need

$$\dim(\rho_1 \otimes \rho_2) = \dim(\rho_1)\dim(\rho_2)$$

$$\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2)$$

VN entropy fails this, but we do have

$$\dim_q(\rho) = \mathrm{Tr}(\hat{\rho}^q)$$

So look at

$$\chi_q(\rho) = \sum_{k=0}^{|I|} (-1)^k \left[\sum_{|T|=k} Tr[\hat{\rho}_T^{\,q}] \right] \in \mathbb{R}$$

$$= (1-q) \sum_{\phi \subseteq T \subseteq I} (-1)^{|T|} S_q^{Tsallis}(\hat{\rho}_T)$$

Where

$$S_q^{Tsallis}(\hat{\rho}) = \frac{1}{1-q}\left(1 - Tr[\hat{\rho}^q]\right)$$

$$\text{“} = Tr[\hat{\rho} \log_q \hat{\rho}^{-1}] \text{”}$$

Note: as $q \longrightarrow 1$, $S_q^T(\hat{\rho}) \longrightarrow S_{vn}(\hat{\rho})$

So

$$\chi_q(\rho) \xrightarrow{q \to 1} 0$$

$$\frac{d}{dq}\chi_q(\rho) \;\text{ or }\; \frac{1}{1-q}\chi_q(\rho) \xrightarrow{q \to 1} I(\hat{\rho})$$

as $q \longrightarrow 0$ we have

$$\chi_q(\hat{\rho}) \xrightarrow{q \to 0} \sum_{T \subseteq I} (-1)^{|T|} \operatorname{rank}(\hat{\rho}_T) \in \mathbb{Z}$$

# of irreducible reps in $G\hat{\rho}_T$.

To recover $\mathbb{R}$-version use some C^*-algebra Theory:

$$\text{Powers of Relative Modular Ops } \left(\Delta_{\hat{p},\hat{q}}\right)^{q} : G_{\hat{q}} \longrightarrow G_{\hat{q}}$$

q ← Power

$\hat{p} \lesssim \hat{q}$

$$\Big\downarrow$$

Chain-Complex Automorphisms

$$\Big\downarrow \text{ Lefschetz-Index}$$

$$\chi_q(\hat{p} \,\|\, \hat{q})$$

Finite dimensions: Take $\hat{q} = \mathbb{1}_n$

(or $\hat{q} = \frac{1}{n}\mathbb{1}_n$ to recover χ up to a constant)

<u>Rmks</u>:

- GHZ State:

$$\psi = \frac{1}{\sqrt{2}}\left(|0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle\right)$$

$$\chi^{q}_{\hat{p}_\psi} \equiv 0$$

But

$$EH^{(A,B,C)} \cong \mathbb{C}^7[0] \oplus \mathbb{C}^7[1]$$

(• Bell State: $\psi = (|00\rangle + |11\rangle)/\sqrt{2}$

$$\chi^{q} \equiv 4 - 2^{q+1}$$)

- Sophisticated version: $H^\bullet$ is a module for a graded algebra

  or $H^\bullet$ is a "left Hilbert algebra" ⇝ "Massey Products."