The Joy of Watching Your BPS States Grow ← (Both in Size and number)

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Main Physical Results (w/ D. Galakhov, P. Longhi, G. Moore, A. Neitzke)

1. At certain regions of vacuum moduli, 4-dim. $\mathcal{N}=2$ SU(3) SYM has a Density of States (DOS) that grows exponentially with the mass. ← Growth contributed by BPS states ← ((Usually) Stable against variance of moduli)

2. The BPS Indices (a weighted count of BPS States) in a large class of $\mathcal{N}=2$ F.T. (Theories of Class $S[A_{K-1}]$) are determined by algebraic equations.

Naively: Alg. Eqns. $\rightsquigarrow$ Exponential Growth in DOS is "generic."

Why is this interesting?

1. Exponential DOS in Field theory seems impossible. (naively)
 (via Std. Thermodynamical Arguments)

2. Exponential Growth of DOS ⇒ "Hagedorn" Limiting temps
 ⇒ Possible phase transitions.

3. In SU(3) SYM there may be ∞^{ly} many Hagedorn temps

Low Expectations: Exponential Growth is impossible

Start w/ a UV-Complete Field theory (e.g. asymptotically Free)

=> Large Energy Phenomena are well-described by a CFT.

(only really need scale-inv.)

(d-1, 1) - CFT in a box OF Volume V; heated up to a temp T:

Dim^l Analysis =>

Energy in box $\longrightarrow$ $$E(V,T) = \alpha \cdot V \cdot T^{d}$$

Entropy (dimensionless) $\longrightarrow$ $$S(V,T) = \beta \cdot V \cdot T^{d-1}$$

Thus

$$S(E,V) = \text{Const.}\; V^{1/d} E^{(d-1)/d}$$

=>

$$\#\,\text{States at energy } E \text{ in CFT} = K \cdot e^{S(E,V)} = K \cdot e^{c V^{1/d} E^{(d-1)/d}}$$

Thus,

$$\#\,\text{States at energy } E \text{ in Theory} \overset{E\to\infty}{\sim} K e^{c V^{1/d} E^{(d-1)/d}}$$

Sub-exponential in E!

$N=2$, $d=4$ Field Theory

$\underline{N=2}$: • Hilbert Space is a rep$^{\underline{n}}$ of $N=2$ Super-Poincaré Algebra "$P^{4|8}$"

$$\text{Super-Poincaré} = \text{Poincaré} + 8 \text{ odd susy} + \text{Complex-Valued Central elt. } \hat{Z}$$

- Reps of $P^{4|8}$ $\longleftrightarrow$ $M^2 > 0$; $I^2 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$; $Z \in \mathbb{C}$

- BPS Bound: $M \geq |Z|$

- BPS rep : 1-particle irrep w/ $M = |Z|$.

  4 out of 8 susy are rep$^{\underline{n}}$ trivially, which 4 is encoded in the phase $\mathrm{Arg}(Z)$. "Stable" as we deform the theory.

Low Energy EFT : Typically $U(1)^r$ abelian gauge theory

(Low Energy) Data

- $\mathcal{B}$: Coulomb branch (Space of Vacua)

  $\mathrm{rk}\,\Gamma_u = 2r + \#\mathrm{Flav.}$

- $\hat{\Gamma} \longrightarrow \mathcal{B}$ local sys. of possible electric/mag./Flavour Charges

  $+$

  Pairing $\langle \cdot , \cdot \rangle_u : \hat{\Gamma}_u \times \hat{\Gamma}_u \longrightarrow \mathbb{C}$

- $Z_u : \hat{\Gamma}_u \longrightarrow \mathbb{C}$ Central Charge Linear Function

  (EM Charge $\Rightarrow$ Known Central Charge)

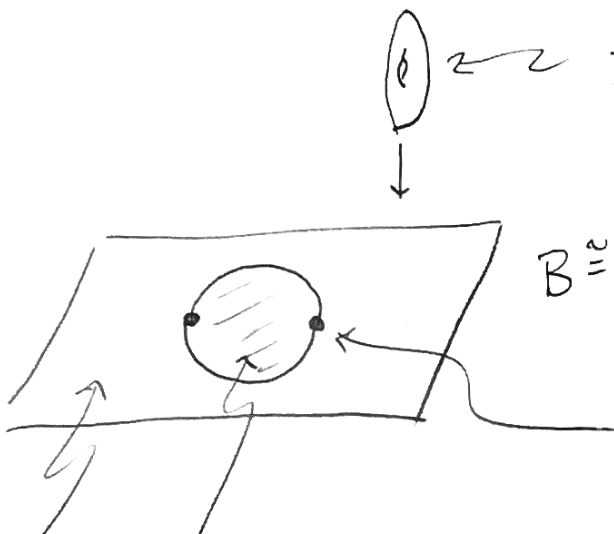
Define the index:

$$\Omega(\gamma; u) = \text{Weighted Count of BPS States in Theory } T_u \,,\ u \in \mathcal{B}$$

Then $\Omega(\gamma, u)$ is Piecewise Stable as we vary $u \in \mathcal{B}$.

Ex: $SU(2)$ SYM (c.f. Seiberg-Witten)

$\Gamma_u = H_1(\Sigma_u; \mathbb{Z})$



$\mathcal{B} \cong \mathbb{C}$

Points where BPS dyons become massless

∞'ly many BPS States

Two BPS dyon-hypermults.

$$\Omega(\gamma_1; u) = \Omega(\gamma_2; u) = +1$$

$$\Omega(\gamma_i; u) = +1$$

$$\Omega(\gamma_{vm.}; u) = -2$$

Jumping Behaviour Well-Understood!

$$\underset{\text{at point } u_*}{\Omega(\gamma; u_*)} \xrightarrow[\text{Gaiotto - Moore - Neitzke}]{\text{Kontsevich - Soibelman}} \Omega(\gamma; u) \quad \forall u \in \mathcal{B}$$

+ Known Walls.

Remark:

$$4\,\Omega(n\gamma) \le \#\text{ BPS States of Charge } n\gamma \le \#\text{ States of mass } M = |n||Z_\gamma|$$

So naively $\Omega(n\gamma)$ can grow at most as fast as $K \cdot e^{c \cdot n^{3/4}}$.

Techniques For Computing Ω:

- BPS Quivers (requires partial knowledge of spectrum at u_*)
- Spectral Networks (Theories of Class S).

m- Kronecker Quiver

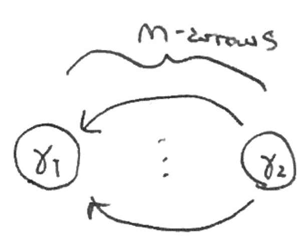
Suppose T_u contains two BPS hypers:

$$\exists \gamma_1, \gamma_2 \quad w/ \quad \Omega(\gamma_1; u) = \Omega(\gamma_2; u) = +1$$

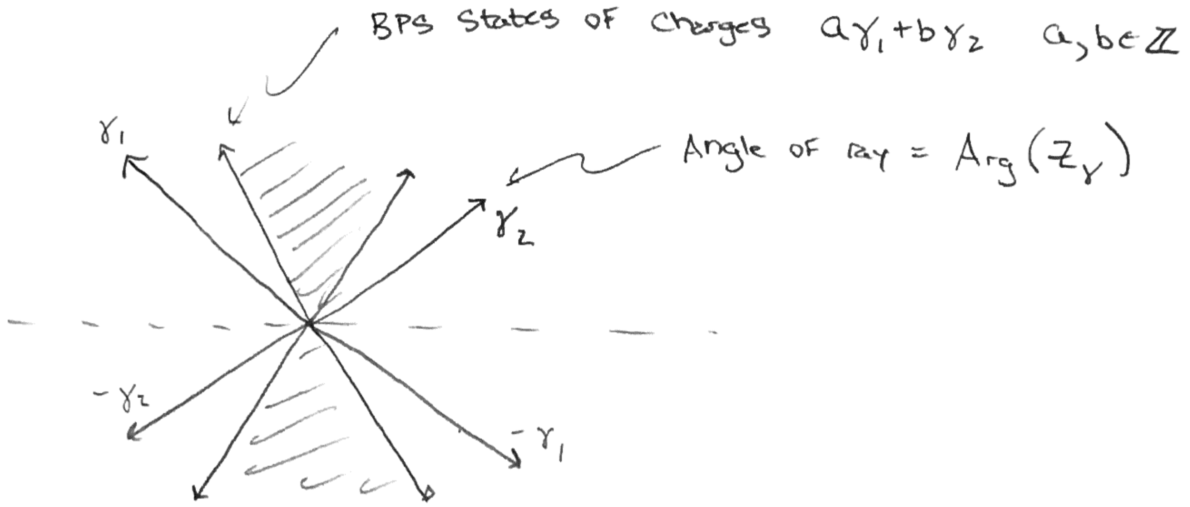
and

$$\langle \gamma_1, \gamma_2 \rangle = m > 0$$

$$\updownarrow$$

BPS Quiver 

Using Quiver Technology: Dense Arc of BPS states!



$\gamma = \gamma_1 + \gamma_2 \Rightarrow$

$$\Omega(n\gamma) \overset{n\to\infty}{\sim} (-1)^{mn+1} K \cdot n^{-5/2} e^{c_m \cdot n}$$

$$\left(\begin{array}{l} c_m = (m-1)^2 \log\left[(m-1)^2\right] - m(m-2) \log\left[m(m-2)\right] \\ K_m = \frac{1}{m-1} \sqrt{\frac{m}{2\pi(m-2)}} \end{array} \right)$$

G. Moore: No WWC (cannot cross walls to get Kronecker m-quiver)

Reality: Nope, SU(3) SYM:

Strong Coupling: 6-hypers

$\Big\downarrow$ Wall-Crossing

m-Kronecker Subquiver.

Spectral Networks

Theories of Class $S[A_{K-1}]$

Riemann Surf. C
\+ (decorated) punctures $\rightsquigarrow$ $\mathcal{N}=2$, $d=4$ Field Theory

E.g. $C = \mathbb{CP}^1$ w/ irreg. Punc at $0,\infty$, $K=3$ $\rightsquigarrow$ $SU(3)$ SYM (pure)

Spectral Networks: Decorated Graphs on C

Point $u \in B$, $\vartheta \in S^1$ $\rightsquigarrow$ System of ODEs whose integral curves give S.N.

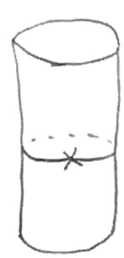
BPS State w/ charge γ s.t. $\operatorname{Arg} Z_\gamma = \vartheta$ $\longleftrightarrow$ Degenerate S.N. at phase ϑ.

$\longleftrightarrow$ BPS states of Charge $n\gamma$ (off of Wall)

E.g

1) Hypermultiplet

2) Vectormultiplet



S.N. Machinery

$$\Omega(n\gamma) \longleftrightarrow T_\gamma = \prod_{n\geq 1} (1-(\pm 1)^n z^n)^{n\,\Omega(n)}$$

Degen. Spectral Network $\rightsquigarrow$ Algebraic Eqn. For T_γ

E.g.:

- "m-herd" network $\rightsquigarrow$ $P - zP^{(m-1)^2} - 1 = 0$

  $\updownarrow$ $\qquad$ $T_\gamma = P^m$

  Charge $n\gamma = n(\gamma_1+\gamma_2)$ States For m-Kronecker

  Predicted by KS, proven by Reineke

  $\updownarrow$

  Exponential Degeneracies!

- $(3,2|m)$-herd: 39^{th} deg. poly.

  $\updownarrow$

  Charge $n(3\gamma_1+2\gamma_2)$ States

<u>Note</u>: BPS indices coming from alg. eqns. (Finite, deg. S.N.) have asymptotics of the form:

$$\Omega(n\gamma) \sim C \cdot n^\alpha \cdot \sum_{i=1}^{\ell} \left(\frac{1}{\rho_i}\right)^n$$

Where $\alpha \in \mathbb{Q}$, $\rho_i \in \overline{\mathbb{Q}} \subset \mathbb{C}$ are s.t. $|\rho_i| \leq 1$.

Generic alg. eqn. $\Rightarrow$ $\Omega(n\gamma) \sim C \cdot n^{-5/2} \left(\frac{1}{\rho}\right)^n$

with $\rho \in (-1,1) \cap \overline{\mathbb{Q}}$.