Quantum Chern Simons

Recall: Classical Chern-Simons

Setup.

- X^3 a 3-manifold
- G Compact, 1-connected, Lie group
- $\text{Conn}_G(X) = \left\{ (P, \Theta) : \begin{array}{c} P \\ \downarrow \\ M \end{array} \text{ Princ } G\text{-bund.}, \ \Theta \in \Omega^1(P; \mathfrak{g}) \text{ Connection} \right\}$

Then the Chern-Simons action is

$$CS : \text{Conn}_G(X) \longrightarrow \mathbb{R}/\mathbb{Z}$$

$$: (P, \Theta) \longmapsto \int_{X^3} s^* \alpha_{cs}(\Theta) \quad \text{mod } 1$$

where

- $s : X \to P$ is a section (every princ. G-bundle is trivializable) Cpt, 1-con. G
- $\alpha_{cs}(\Theta) = \langle \Theta \wedge F_\Theta \rangle - \frac{1}{6} \langle \Theta \wedge [\Theta \wedge \Theta] \rangle \in \Omega^3(P)$

  is the Chern-Simons 3-form

Remarks

- CS is independent of choice of s when taken mod 1
- α_{cs} depends on a choice of ad-invariant pairing on $\mathfrak{g}$

$$\updownarrow$$

Choice of $W \in H^4(BG; \mathbb{Z})$ "Quantization Cond."

In this talk:

$$G = SU(n)$$

$$H^4(BG; \mathbb{Z}) \cong \underset{\substack{\rotatebox{90}{$\in$} \\ k}}{\mathbb{Z}} \longleftrightarrow k \, \text{Tr}_R(\cdot, \cdot) \qquad R = \text{defining rep}$$

Note

- CS is invariant under gauge transf.
- $\mathrm{Crit}(CS)$ = Flat principal G-bundles on X^3
- w/ $\partial X^3 \neq 0$ CS is invariant under gauge transf that restrict to $1_{P|_{\partial X}}$ on ∂X.

Claim

Path Integral Quant. $\rightsquigarrow$ I. Functor

$Z: \mathrm{Bord}_{\langle 2,3\rangle} \longrightarrow \mathrm{Vect}$

(1) $Z(X^3 \text{ closed}) = \#$

(2) $Z(\partial X) = \mathcal{H}_{\partial X}$

Easiest to see under hol quant

II. Knot invariants

Canonical / hol. quant.

I.

(1) The path integral for $\partial X = \phi$

$$Z(X) = \int_{\mathcal{C}_X} D_{\mathcal{C}_X} \Theta \, e^{ikCS(\Theta)} \in \mathbb{C}$$

$\left(\text{“=”} \int_{\pi_0 \mathcal{C}_X} D[\Theta] e^{ikCS[\Theta]} \right)$

Groupoid of Connections

Remark: Measure is not rigourously defined

Q: How to actually integrate?

Step 1: Fadeev-Popov Method / Equivariant Localization

- Integration over groupoid/quotient is hard

$$\int_{\mathcal{C}_X} \mathcal{D}_{\mathcal{C}_X}\Theta \, e^{iKCS[\Theta]} \rightsquigarrow \int_{\mathrm{Conn}(X)} \mathcal{D}\Theta \, \mathcal{D}c \, \mathcal{D}\bar{c} \, e^{iKCS'[\Theta, g, c, \bar{c}]}$$

Conn(X) ← Affine Space

- Implicit Choice of "gauge slice": $\mathrm{Conn}(X)/\hat{\mathcal{G}} \hookrightarrow \mathrm{Conn}(X)$

<u>Requires a metric g</u>

- $c, \bar{c}$: ghosts: Fields valued in odd vector space

Step 2: ∞-dim$^{\underline{l}}$ Stationary phase Approximation in $K \to \infty$ limit ($\hbar \to 0$ limit)

Ex: Finite dimensions on $\mathbb{R}^n$: f a morse function $f: \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} dx_1 \cdots dx_n \, g(x) \, e^{iKf(x)}$$

$$\overset{K\to\infty}{\sim} \sum_{y \in \mathrm{Crit}(f)} g(y) \, e^{iKf(y)} \cdot \frac{C_n \, e^{\frac{i\pi}{4}\sigma_y(f)}}{K^{n/2}\sqrt{|\det(\mathrm{Hess}_f(y))|}} + \mathcal{O}\left(\frac{1}{K^{n/2+1}}\right)$$

Asymptotic Series → $\sim$

$\sigma_y(f)$ ← Signature of Hessian

For our Situation:

$$f \rightsquigarrow CS'$$

$$\mathrm{Crit}(CS') \cong \mathrm{Crit}(CS)/\hat{\mathcal{G}}$$

(as sets / Symplectic manifolds away from Sing.)

$$\mathrm{Crit}(CS)/\hat{\mathcal{G}} = \mathcal{M}_X \overset{\text{Set}}{\cong} \mathrm{Hom}(\pi_1(X), G)/G$$

$\mathcal{M}_X$ ← Moduli-Space of Flat Conn. on X

Assume $\dim(\mathcal{M}_X) = 0$ (e.g. X connected and $H_1(X;\mathbb{Z}) = 0$)

Then

$$Z(X^3) = \sum_{\theta \in \mathcal{M}_X} \mu(\theta)$$

Where

$$\mu(\theta) = e^{iK\,CS(\theta)} \cdot RS(\theta) \cdot e^{\frac{i\pi}{2}\eta_g(\theta)}$$

Ray - Singer Torsion :
Indep of metric $\Rightarrow$ Diffeomorphism Invar

Still a Problem $\eta_g(\theta)$ depends on choice of metric g

Fix : Wrt some trivialization of (P, θ) :

1) $\eta_g(\theta) = \eta_g(0) + \underbrace{\text{Const}}_{\frac{C_2(G)}{\pi}} \cdot CS(\theta) \mod 1$

Trivial Connection wrt trivialization
Depends on metric

RHS Indep. of triv. up to $\mathbb{Z}$.

2) Atiyah : $\eta_g(0) \rightsquigarrow$ determined solely by a choice of 2-Framing

$$e^{\frac{i\pi}{2}\eta_g(0)} = \exp\left\{ \frac{2\pi i \dim(G)\, \sigma(\alpha)}{8} \right\}$$

α a 2-Framing (Trivialization of Spin(6)-bundle assoc. to Frames on $TX \oplus TX$)

- $\exists$ a Canonical 2-Framing

$\Rightarrow$

$$Z_{old} \rightsquigarrow Z_{new} \leftrightsquigarrow K \mapsto K + \frac{C_2(G)}{2}$$

Defines a Topological invariant (in the large K limit)

(2) X^3 w/ ∂

Rmk: • With ∂ the "phase factor"

$$e^{ikCS[\Theta, s]}$$

depends on the choice of trivialization $s : X \longrightarrow P$, but in an understood way:

$$e^{ikCS[\Theta]} \in \mathcal{L}_{[\Theta]_{\partial X}}$$ ← (hermitian) line (depends on $\Theta|_{\partial X}$).

Section s ⟿ Iso $\mathcal{L}_{[\Theta]_{\partial X}} \xrightarrow{\sim} \mathbb{C}$

• Path integral depends on the choice of ∂-condition;

$$\mathcal{C}_X \xrightarrow{\text{rest.}} \mathcal{C}_{\partial X} \xrightarrow{\pi_0} \mathrm{Conn}(\partial X)/\hat{G}_{\partial X}$$

$\| \leftarrow$ as sets

$$\mathcal{M}_{\partial X}$$

So

Actually a subset of

$$Z_{X^3} : \eta \in \text{"}\mathcal{M}_{\partial X}\text{"} \longmapsto \int_{\Gamma_\eta} D\Theta \, e^{ikCS[\Theta]}$$

$\Gamma_\eta = \mathrm{rest}^{-1}(\eta)$, $\quad e^{ikCS[\Theta]} \in \mathcal{L}_\eta$

$$\Rightarrow \quad Z_{X^3} \in \Gamma\left(\begin{array}{c}\mathcal{L}\\ \downarrow \\ \mathcal{M}_{\partial X}\end{array}\right)$$

So

X^3 ⟿ $\mathcal{H}_{\partial X}$ = Sections of some line bundle over $\mathcal{M}_{\partial X}$

Holomorphic Quantization (Easily Seen For $\partial X \times [0,1]$)

Classical Theory w/ Kähler manifold (M, J, ω) Of "initial Conditions"/ ∂-Conditions $\xrightarrow{\text{Quantize}}$ $\mathcal{H} = \Gamma_{hol.}\begin{pmatrix} \mathcal{L} \\ \downarrow \\ M \end{pmatrix}$

w/ $c_1(\mathcal{L}) = [\omega] \in H^2(M; \mathbb{Z})$

Roughly Indep of Choice of J up to proj. unitary transf.

Claim:

- $\mathcal{M}_{\partial X}$ is Symplectic $\leftarrow$ Initial Conditions For Classical Theory on $\partial X \times [0,1]$

  Choice of $\mathbb{C}$-Str. on ∂X $\rightsquigarrow$ $(\mathcal{M}_{\partial X}, \omega, J)$ Kähler

  $\Rightarrow$ hol. quant., level $K \longleftrightarrow \mathcal{L}^{\otimes K}$

- Verlinde: $\dim \mathcal{M}_{\partial X}^{(K)} < \infty$

  Heuristic: $\dim \mathcal{H}ilb \sim \mathrm{Vol}(M, \omega)$

  Witten provides an estimate For $\mathrm{Vol}(M, \omega)$

  ($\cdot$ Ex: $G = \mathbb{T}$, $\mathcal{M}_{\partial X} \cong \mathbb{T}^{2g_{\partial X}}$, $\dim(\mathrm{Hilb}_K) = K^g$) Θ-Functions

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Functoriality: $X = X_1 \cup_\varphi X_2$ $\quad \varphi : \partial X_1 \longrightarrow \overline{\partial X_2} \in \mathrm{Diff}^+(\partial X_1, \partial X_2)$

Composition of bordisms

$$\Rightarrow \quad Z(X) = \langle Z(X_1), \varphi^* Z(X_2) \rangle \quad (\mathcal{H}_{\partial X_1} \cong \mathcal{H}^*_{\partial X_2})$$

Induced action on Hilbert Space

$$\Rightarrow \quad Z_{CS} : \mathrm{Bord}_{\langle 2,3 \rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

Knots

Setup

- X^3 manifold (w/o ∂)
- $L \subset X^3$ an oriented link w/ Framing of ν_L $\quad L = \coprod_{i=1}^{r} C_i$
- R_i G-reps associated to each C_i

Define:

$$W_{R_i}(C_i) : \mathrm{Conn}(M) \longrightarrow \mathbb{C}$$
$$\Theta \longmapsto \mathrm{Tr}_{R_i}[\mathrm{hol}_\Theta C_i]$$

Then we can modify the path integral to calculate invariants of 3-man. in the presence of knots/links:

$$\langle L \rangle := Z(X^3, L) = \int_{\mathrm{Conn}(M,L)} D\Theta\, e^{iK\, CS[\Theta]} \prod_{i=1}^{r} W_{R_i}(C_i)$$

Claim:

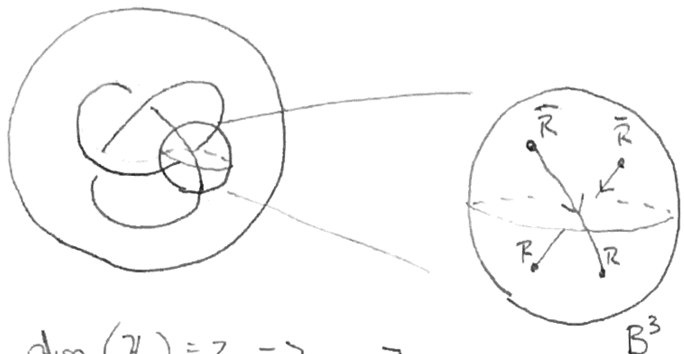
- $X^3 = S^3$, $G = SU(2)$, $L = C$, $R =$ Fund$^{\underline{l}}$ Rep.

  $\Rightarrow Z(S^3, C) = \longleftrightarrow$ Jones poly. for C

Sketch of proof:

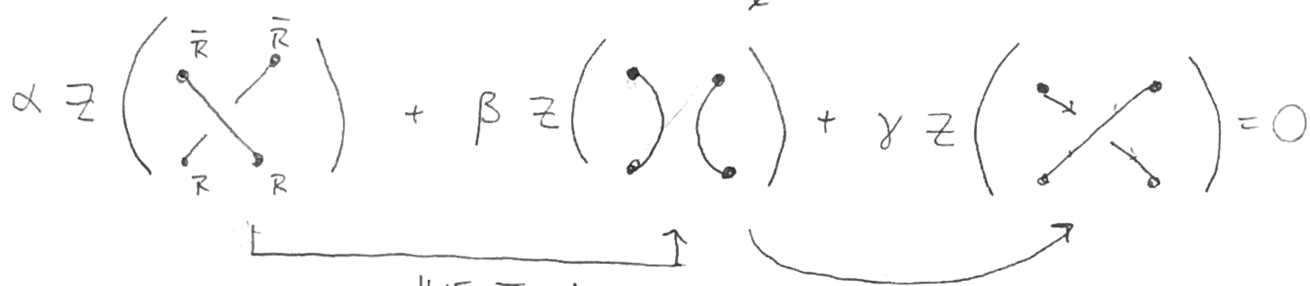
Use surgery along with the fact that $\dim(\mathcal{H}_{S^2, R, \bar{R}, R, \bar{R}}) = 2$

Surgery:



$\dim(\mathcal{H}) = 2 \Rightarrow \quad \exists\, \alpha, \beta, \gamma$ s.t.

Skein Relation



Half Twist

$K \in \mathrm{Diff}^+(S^2, 4 \text{ pts.})$

Half twist

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Moore-Seiberg: Determine induced action of K on $\mathcal{H}$

$\Rightarrow$

$$\alpha = -\exp\left(\frac{\pi i N}{N+K}\right), \quad \beta = \exp\left(\frac{\pi i}{N+K}\right) - \exp\left(\frac{-\pi i}{N+K}\right)$$

$$\gamma = \exp\left(\frac{-\pi i N}{N+K}\right)$$

$q := \exp\left(\frac{2\pi i}{N+K}\right) \Rightarrow$