Clifford Algebras and Dirac Operators

Motivation

Consider Euclidean space $\mathbb{E}^n$ equipped w/ orthog. Coords $x^1, \ldots, x^n$. Then we have the Laplace Op.

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2}$$

Q: Does there $\exists$ a 1^{st} order diff$^{\underline{l}}$ op. D w/ $D^2 = \Delta$? (Can we split $\Delta f = 0$ into 1^{st} order eqns.)?

Ansatz:

$$D = \gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \cdots + \gamma^n \frac{\partial}{\partial x^n}$$

Then

$$D^2 = \Delta \iff \begin{cases} (\gamma^i)^2 = -1 \\ \gamma^i \gamma^j + \gamma^j \gamma^i = 0, \ i \neq j \end{cases} \longleftrightarrow \underbrace{\gamma^i \gamma^j + \gamma^j \gamma^i = -g^{ij}_{\mathbb{E}^n}}_{O(n) \text{ invariant}}$$

Thus, we want to construct an algebra A with a map $V \overset{\varphi}{\hookrightarrow} A$, $V \cong \mathbb{R}^n$, s.t.

φ v.s. morphism $\Rightarrow$ equiv. conditions

$$\varphi(v)^2 = -(v, v) \cdot 1$$

$$\varphi(v)\varphi(w) + \varphi(w)\varphi(v) = -2(v, w) \cdot 1 \qquad (*)$$

Solution: Take the Free algebra over V subject to the relation $(*)$.

Def

Let V be a v.s. over $K = \mathbb{R}$ or $\mathbb{C}$ equipped with a Symmetric bilinear Form $(\cdot,\cdot)$, and

$$\otimes V := K \oplus \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

Then

$$\mathrm{Cliff}(V) := \otimes V \Big/ (v \otimes v + (v,v) \cdot 1)$$

is an alg. over K.

Def

A Clifford Algebra for V is a pair (A, φ) where

(1) A is a unital algebra

(2) $\varphi : V \hookrightarrow A$ is a map of v.s. s.t. $\varphi(v)^2 = -(v,v)\cdot \mathbb{1}$

(3) The pair (A,φ) satisfies the following universal prop:

If (A', φ') is another pair satisfying (1) and (2) then $\exists !$ alg. morphism η s.t.

$$\begin{tikzcd} V \arrow[r, hook, "\varphi"] \arrow[dr, "\varphi'"'] & A \arrow[d, dashed, "\eta"] \arrow[dl, phantom, "\circlearrowright"] \\ & A' \end{tikzcd}$$

Prop

$\mathrm{Cliff}(V)$ is a Clifford Alg. and is unique up to iso. of algebras

Proof:

Need only check the universal prop. to show uniqueness. $\boxtimes$

Remarks

- $(\cdot,\cdot) \equiv 0 \Rightarrow \mathrm{Cliff}(V) = \otimes V / (v \otimes v) = \Lambda^\bullet V$

- In general $\mathrm{Cliff}(V) \underset{V.S}{\cong} \Lambda^\bullet V$ as vector spaces $\Rightarrow \dim \mathrm{Cliff}(V) = 2^{\dim(V)}$.

Better:

Degree Filtration on $\otimes V \rightsquigarrow$ Filtration on $\mathrm{Cliff}(V)$
$F_0 \subset F_1 \subset \cdots \subset F_n$

Then

$$\mathrm{Gr}_q \mathrm{Cliff}(V) = F_q / F_{q-1} \cong \Lambda^q V$$

$\Rightarrow$

$$\underset{\bigoplus \mathrm{Gr}_q \mathrm{Cliff}(V)}{\overset{\|}{\mathrm{Gr}_\bullet \mathrm{Cliff}(V)}} \cong \Lambda^\bullet V \quad \text{as algebras.}$$

Dirac Operators

Fix V a $\mathbb{R}$ v.s. with inner product; let S be a v.s. over $K = \mathbb{R}$ or $\mathbb{C}$ that is also a left module for $\mathrm{Cliff}(V)$.

$$\mathrm{Cliff}(V)\ \text{Action} \longleftrightarrow \begin{array}{c} C : V \longrightarrow \mathrm{End}_K(S) \\ \mathbb{R}\text{-linear and } C(v)^2 = -(v,v)\cdot \mathbb{1}_S \end{array}$$

Remark

In the $K = \mathbb{C}$ case we can extend the action to $\mathrm{Cliff}_{\mathbb{C}}(V) := \mathrm{Cliff}(V) \otimes_{\mathbb{R}} \mathbb{C}$.

Example

$\Lambda^{\cdot} V$ is a $\mathrm{Cliff}(V)$-mod when equipped with

$$C(v) = \epsilon(v) + \iota(v)$$

Where

$$\epsilon(v)\, w = w \wedge v$$

and

$$(v, \iota(\xi) w) = (\epsilon(\xi) v, w)$$

$$\begin{cases} \iota(v) w = \langle v, w \rangle, \quad w \in \Lambda^1 V \\ \iota(v)(w_1 \wedge w_2) = (\iota(v) w_1) \wedge w_2 + (-1)^{|w_1|} w_1 \wedge \iota(v) w_2 . \end{cases}$$

Exercise: Check $C(v)^2 = [\epsilon(v), \iota(v)] = -(v,v)$.

$\Lambda^{\cdot} V \otimes \mathbb{C}$ is a $\mathrm{Cliff}_{\mathbb{C}}(V)$-mod.

Let $e_1, \dots, e_n$ be a basis for V, then we define the Dirac Operator

$$D(\cdot) = \sum_{i=1}^{n} \underbrace{c(e_i)}_{\gamma^i \text{ in motivation}} \cdot \left[\frac{\partial}{\partial x^i} (\cdot) \right] : C^\infty(V; S) \longrightarrow C^\infty(V; S)$$

Then $\forall s \in S$,

$$\begin{aligned} D^2 s &= \sum_{i,j} c(e_j)\, \partial_j \left[c(e_i)\, \partial_i s \right] \\ &= \sum_{i,j} c(e_j) c(e_i)\, \partial_j \partial_i s \\ &= \sum_i c(e_i)^2 \partial_i^2 s + \sum_{i<j} \left[c(e_i) c(e_j) + c(e_j) c(e_i) \right] \partial_i \partial_j s \\ &= -\sum_i \partial_i^2 s \end{aligned}$$

Note: We can think of the above as a Dirac Op. For the trivial bundle

$$\begin{array}{c} V \times S \\ \downarrow \\ V \end{array}$$

with trivial connection $\nabla = dx_i \wedge \frac{\partial}{\partial x_i}$. How do we generalize?

Def

Let S be a bundle of Clifford modules over a Riemannian manifold M. S is a Clifford bundle if it is equipped with a hermitian metric $h(\cdot,\cdot)$ and compatible connection ∇^S s.t.

(1) The Clifford action of $Cliff(T^*_m M)$ is skew-adjoint $\forall m$, i.e. $\forall \alpha \in T^*_m M$ and $s_1, s_2 \in S_m$

$$h_m(c(\alpha)\cdot s_1, s_2) + h_m(s_1, c(\alpha)s_2) = 0$$

(γ^i are skew-adjoint matrices)

(2) ∇^S is compatible with the Levi-Civita connection on T^*M:

$$\nabla^S[c(\alpha)s] = c[\nabla^M(\alpha)]s + c(\alpha)\nabla^S s$$

where $\alpha \in \Gamma(T^*M)$, $s \in \Gamma(S)$ and

$$\nabla^M : \Gamma(T^*M) \longrightarrow \Gamma(T^*M^{\otimes 2})$$

is the Levi-Civita connection

Def

The Dirac operator D of a Clifford bundle S is the 1^{st} order op:

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(S \otimes T^*M) \xrightarrow{c} \Gamma(S)$$

Let $\alpha_i : \underset{M}{\overset{\cap}{U}} \longrightarrow T^*M$ be a local orthonormal frame, then

$$Ds = \sum_i c(\alpha_i)\nabla_i s$$

Weitzenböck / Lichnerowicz Formula

We wish to compute D^2 in our more general situation.

Consider local "Synchronous" Coordinates at a point $m \in M$:

$$e_i : U_m \longrightarrow TM$$

$$\alpha_i : U_m \longrightarrow T^*M \quad \text{(dual Frame)}$$

with

$$\nabla_i \alpha_j = 0 \quad \text{(at } m)$$

$$[e_i, e_j] = 0 \quad \text{(at } m)$$

then

$$D^2 s = \sum_{i,j} c(\alpha_j) \nabla_j [c(\alpha_i) \nabla_i s]$$

$$= \sum_{i,j} c(\alpha_j) c(\alpha_i) \nabla_j \nabla_i s$$

$$= -\sum_i \nabla_i^2 s + \sum_{j<i} c(\alpha_j) c(\alpha_i) [\nabla_j \nabla_i - \nabla_i \nabla_j] s$$

$$= -\sum_i \nabla_i^2 s + \underbrace{\sum_{j<i} c(\alpha_j) c(\alpha_i) \Omega_\nabla(e_j, e_i)}$$

K: Clifford Contraction of the Curvature

$K \in \mathrm{End}(S)$: 0^{th} order operator

Remark

$-\nabla_i^2$ is the Synchronous Coordinate expression for the Laplacian $\nabla^* \nabla$ where

$$\nabla^* : \Gamma(E \otimes T^*M) \longrightarrow \Gamma(E)$$

is the formal adjoint of ∇, defined by

$$h(\nabla s, r)_{S \otimes T^*M} = h(s, \nabla^* r)_S + d(\Omega_{n-1}) \qquad \begin{array}{l} \forall s \in \Gamma(S) \\ \forall r \in \Gamma(S \otimes T^*M) \end{array}$$

(i.e. For L^2 Sections:

$$\langle \nabla s, r \rangle = \langle s, \nabla^* r \rangle \quad \text{For} \quad \langle \cdot , \cdot \rangle = \int h(\cdot , \cdot) \, \mathrm{Vol} \;)$$

Thus, in Coordinate-Free notation

$$D^2 = \nabla^* \nabla + K : \Gamma(S) \longrightarrow \Gamma(S)$$

Thm (Bochner): M compact manifold.

If $K_m \in \mathrm{End}(S_m)$ has least eigenvalue >0 $\forall m \in M$, then there are no non-trivial Sol^{n}s to $D^2 s = 0$.

Proof

Compactness + Eigenvalue Cond. $\Rightarrow$ $\langle Ks, s\rangle \geq C\|s\|^2$ For some $C>0$. But, by Weitzenböck,

$$\langle Ks, s\rangle = \langle D^2 s, s\rangle - \|\nabla s\|^2 \leq 0 \quad ↯$$

∎

Refinement of Weitzenböck

Prop

The curvature Ω_∇ of a Clifford bundle S can be written as

$$\Omega_\nabla = R^S + F^S \in \Omega^2(\mathrm{End}(S))$$

Where

Riemann curv.

$$R^S(X,Y) = \frac{1}{4} \sum_{k,l} c(\alpha_k)\, c(\alpha_l) \left(R(X,Y) e_k, e_l\right)$$

and F^S Commutes with the action of the Clifford Alg:

$$[F^S(X,Y), c(z)] = 0$$

F^S is called the twisting curvature.

Corollary

Vanishes for spin-bundles

Riemann Curv.

$$D^2 = \nabla^* \nabla + F^S + \frac{1}{4} R \cdot \mathbb{1}_S$$

Then For $p_1, p_2 \in P$; $q_1, q_2 \in Q$

$$(p_1, p_2) = (q_1, q_2) = 0$$

and $P^* \cong Q$ via inner product.

$\Lambda^\cdot P$ is a $\mathrm{Cliff}_{\mathbb{C}}(V)$ module :

$x \in \Lambda^\cdot P$; $v = p + q \in V_{\mathbb{C}}$ $\Rightarrow$

$$cl(p+q)\,x = \sqrt{2}\,(\epsilon(p)\,x + \iota(q)\,x)$$

<u>Exercise</u>:

$$c(p)^2 = c(q)^2 = 0 \;;\; c(p)c(q) + c(q)c(p) = -2(p,q)$$

<u>Note</u>: $\dim_{\mathbb{C}} \Lambda^\cdot P = 2^n$ while the regular rep $\Lambda^\cdot V_{\mathbb{C}}$ has $\dim = 2^{2n}$.

(3) Bundlize the above construction for M complex, Hermitian,

<u>Prop</u>

IF M is Kähler then $D = \sqrt{2}\,(\bar\partial + \bar\partial^*)$ where

$$\bar\partial : \Omega^{0,k} \longrightarrow \Omega^{0,k+1}$$

$$[\nabla^M, J] = 0.$$

- $\mathrm{Cliff}(\mathbb{R}^1) \cong \mathbb{C}$
- $\mathrm{Cliff}(\mathbb{R}^2) \cong \mathbb{H}$
- $\mathrm{Cliff}(\mathbb{R}^3) \cong \mathbb{H} \oplus \mathbb{H}$