What is a (Formal) Derived Stack?

Motivation: Intersection Theory For Schemes

Classical Algebraic Geometry: Bezout's Thm.

C_1, C_2 two curves in $\mathbb{P}^2$ which are $\pitchfork$; then:

$$\underbrace{[C_1] \cap [C_2]}_{\text{Homological Int.}} = \underbrace{[C_1 \cap C_2]}_{\text{"Geometric Intersection"}} = [C_1 \times_{\mathbb{P}^2} C_2]$$

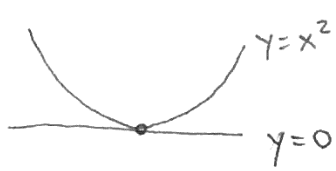
$\cap \in H_0(\mathbb{P}^2; \mathbb{Z})$

Ex: 1) $\{x=0\}, \{y=0\} \subseteq \mathbb{A}^2$ hom int $= 1$

Geom. int:

$$k[x,y]/(x) \otimes_{k[x,y]} k[x,y]/(y) = \underbrace{k[x,y]/(x,y)}_{\text{dimension } 1}$$

2)



$\rightsquigarrow$ hom int $= 2$

Geom. Int:

$$k[x,y]/(y-x^2) \otimes_{k[x,y]} k[x,y]/y = \underbrace{k[x,y]/(x^2)}_{\text{dimension } 2}$$

non-reduced Scheme

reduced part has dim $= 1$,

Serre's mult. Formula: X, Y Subvarieties of $\mathbb{P}^n$ (irreducible) of Complem

W an irreducible Component of $X \times_{\mathbb{P}^n} Y$, then

$$[C_1] \cap [C_2] = \sum_W m_W$$

$$m_W := \sum_i (-1)^i \dim \mathrm{Tor}^i_{\mathcal{O}_{\mathbb{P}^n, W}} (\mathcal{O}_{X,W}, \mathcal{O}_{Y,W})$$

length $\mathcal{O}_{\mathbb{P}^n, W}$

Idea: Further extend notion of Space so that Bezout's thm. holds:

$$\mathrm{Sch}^{red} \subset \mathrm{Sch} \subset d\mathrm{Sch}$$

Intersection $\quad$ derived intersection

$$\begin{array}{ccc} X \times_V Y & \rightsquigarrow & X \times^h_V Y \\ \updownarrow & & \updownarrow \\ \mathcal{O}_X \otimes \mathcal{O}_Y & & \mathcal{O}_X \otimes^{\mathbb{L}} \mathcal{O}_Y \end{array}$$

Ex. $\{x=0\} \cap \{x=0\}$ in $\mathbb{P}^2$; hom. dim $= 1$

Homological int. $= 1$

Geometrically:

$$k[x,y]/x \otimes^{\mathbb{L}}_{k[x,y]} k[x,y]/x = ?$$

resolve

$$k[x,y,\xi] \xrightarrow{d} k[x,y] \; ; \; H_0 = k[x,y]/x$$

$$\xi \longmapsto x$$

So

$$x \longmapsto x$$
$$y \longmapsto y$$

$$k[x,y]/x \otimes^{\mathbb{L}}_{k[x,y]} k[x,y]/x \simeq \underset{\deg -1}{k[y,\xi]} \xrightarrow{d} \underset{\deg 0}{k[y]}$$

$$\xi \overset{d}{\longmapsto} 0$$

$$y \longmapsto y$$

Giving $\quad \chi =$ Virtual "dim" $= 1$.

What is a derived Stack? (Vezzosi: What is)

<u>Roughly</u>: (Functor of points perspective)

$$\begin{tikzcd}
\mathrm{CAlg}_R \arrow[r, "R\text{-Scheme}"] \arrow[dr, "\text{Stack}"] \arrow[ddr, "\infty\text{-Stack}"] \arrow[dd] & \mathrm{Set} \arrow[d, "i"] \\
& \mathrm{Grpd} \arrow[d, "\text{Nerve}"] \\
\mathrm{cdga}^{\leq 0}/s\mathrm{CAlg}_R \arrow[r, "\text{Derived}", "\text{Stack}"'] \arrow[uur, bend right=60, "R\text{-Scheme}"'] & s\mathrm{Set}
\end{tikzcd}$$

(Derived Rings)

Same in Char 0

<u>Derived Schemes?</u>

As locally ringed Spaces: A derived Scheme is a pair $(X, \mathcal{O}_X)$ where $X \in \mathrm{Top}$ and $\mathcal{O}_X$ is a <u>Stack</u> of simplicial comm. rings on X s.t.

(1) $(X, \pi_0(X))$ is a Scheme

(2) $\forall i > 0$, $\pi_i(\mathcal{O}_X)$ is a quasi-coherent sheaf of $\pi_0(\mathcal{O}_X)$-mods.

$\quad\quad \| \; \mathrm{cdga}^{\leq 0} \leftarrow$ "Connective"

<u>Note</u>: $\quad H^{-i}(A)$

$$\mathrm{Sch} \longrightarrow d\mathrm{Sch}$$

$$(X, \mathcal{O}_X) \longmapsto X, \quad \mathcal{O}_X \Rrightarrow \mathcal{O}_X \rightrightarrows \mathcal{O}_X \to \mathcal{O}_X$$

$$\wr$$

$\mathcal{O}_X[0]$ as a cdga

or any other simplicial resolution

$\left(\begin{array}{l} \quad\quad\quad = H_{-i}(\mathcal{O}_X) \\ \underline{\text{Rmk}}: \; \pi_i(\mathcal{O}_X), \; i \geq 1 \text{ higher "nilpotents"} \\ \quad\quad \text{From cdga perspective} \rightsquigarrow \end{array} \right)$

Also want $(X, \mathcal{O}_X) \simeq (X, \mathcal{O}_X')$ if quasi-iso

∞ - Structure on $\mathrm{Cdga}^{\leq 0}$:

As an elt. of Cat_Δ:

Def: $\Delta^n := \mathrm{Spec}\left(k[t_0, \ldots, t_n] / (\sum_i t_i - 1)\right)$

$C^n := \Omega^*_{dR}(\Delta^n) \in \mathrm{cdga}$

Note: $\{0\} \hookrightarrow \Delta^1 \hookleftarrow \{1\} \rightsquigarrow C^1 \rightrightarrows k$

In gen: $\cdots C^2 \overset{\longrightarrow}{\rightrightarrows} C^1 \rightrightarrows k$

Then $\mathrm{Map}(A, B) \in s\mathrm{Set}$

is defined by

$$\mathrm{Map}_n(A, B) := \mathrm{Hom}_{\mathrm{cdga}}(A, B \otimes C^n)$$

(alternatively defined by simplicial localization using model str.) (1) Fibrations are levelwise surj. W.e. are quasi-iso. (2) Everything is fibrant

(Note: $s\mathrm{Alg}_k \underset{\Gamma}{\overset{N}{\rightleftarrows}} \mathrm{cdga}_k^{\leq 0}$ (N left Quillen adjoint)

Quillen equiv if $\mathrm{Char}\, k = 0$)

Cell cdgas: cofibrant objects in cdga

A a cell dga if $\exists$ a filtration

$$k = A_0 \hookrightarrow A_1 \hookrightarrow A_2 \cdots\cdots \hookrightarrow A$$

s.t. $A_{m+1} \cong A_m \otimes k[x_\alpha]$ w/ $d(x_\alpha) \in A_m$.

derived affine schemes $:= (\mathrm{Cdga}^{cf})^{op} = (\mathrm{Cdga}^{f,cf})^{op}$
(∞-cat.)

$$\left(\begin{array}{l}
\text{dAFF : Full } \infty\text{-Subcat of dSch s.t. } \pi_0(X, \mathcal{O}_X) \text{ is affine.} \\
\text{There is an equivalence of } \infty\text{-cats:} \\
\qquad \Gamma : \text{dAFF}^{op} \rightleftarrows \text{Cdga}^{\leq 0} : \text{Spec} \\
\text{where } \Gamma(\mathcal{O}_X) = \underset{\substack{\Uparrow \\ \text{Induced } \infty\text{-Functor.}}}{p_*} , \; p : X \longrightarrow \underset{\substack{\shortparallel \\ \text{Spec } k}}{*}
\end{array}\right)$$

• dSch $\subset$ dSt are ∞-cats.

$$\text{dSt} = \{F : \text{Cdga}^{\leq 0} \longrightarrow \text{sSet} + \underset{\text{wrt étale str}}{\text{descent}}\}$$

Rmk :

$$\begin{tikzcd}
 & \text{Set} \arrow[dr] & \\
F : \text{Cdga}^{\leq 0} \arrow[ur, dashed] \arrow[rr] & & \text{sSet}
\end{tikzcd} \quad \Rightarrow F \text{ is a Scheme}$$

<u>Back to Classical Geometry</u> : <u>Deformation Theory</u>

First : Small Fuzzy Cap

<u>Rmk</u> . TFAE For A a <u>local</u> k-alg. w/ res. Field k.

(1) A artinian (descending chain Cond.)

(2) A F.d. as a k-V.S.

(3) The max$^{\underline{l}}$ ideal of A is F.g. and nilpotent.

<u>Def</u> • Art/k : local artin. alg. w/ res. Field k

• A deformation Functor is a Covariant Functor

$$F : \text{Art}/k \longrightarrow \text{Set} \rightsquigarrow F : \left[(\text{Sch})^{\text{Sm}}\right]^{op} \longrightarrow \text{Set}$$

s.t. $F(k) = pt.$

Idea: $F(k)$ is the object we want to deform, $F(A)$ = iso. classes of defs over A.

Ex: • X a scheme, $Z \hookrightarrow X$ closed sub. $H_{Z,X}(A)$ = defs of Z in X over Spec A

$H_{Z,X}(A)$ = S-flat closed subschemes Z_S of $X \times S$ s.t. fibre over $(\operatorname{Spec} A)^{red}$ is Z.

$$\begin{tikzcd} Z_B = Z_A \times_A \operatorname{Spec} B \arrow[r] \arrow[d] & Z_A \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

• $F: \mathrm{Sch} \to \mathrm{Set}$ a moduli problem, $p \in F(\operatorname{Spec} k)$ a point. Then

$$\Big(\hat{F}_p := F \times_{\operatorname{Spec} k} p \Big)$$ as a functor

$$\hat{F}_p(A) = \{ \alpha \in F(\operatorname{Spec}(A)) : \alpha|_{\operatorname{Spec}(A/m_A)} = p \}.$$

$$= F(A) \times_{F(A/m_A)} \{p\}$$

($F(A/m_A)$ = Spec k)

Formal Derived Stacks: $F: (\mathrm{cdga}^{\leq 0})^{sm} \longrightarrow s\mathrm{Set}$

Small fuzzy crap. ⇝ Aug $\exists A \to k$

Def: $A \in \mathrm{Cdga}^{\leq 0}_{Aug}$ is small if

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq k$$

A_i a square zero extension of A_{i+1} by $k[n_i]$ for some $n_i \geq 0$

Prop: $A \in \mathrm{Cdga}^{\leq 0}_{Aug}$ is small if

(1) $\pi_n A = 0$ for $n < 0$ and $n \gg 0$

(2) $\pi_n A$ is f.d. as a v.s. over k

(3) $\pi_0 A$ is local w/ max$^{\underline{l}}$ ideal m and $k \to \pi_0 A/m$ is an iso.

Recall: The Cotangent Complex:

Kähler-diff^{ls}: In Schemes the Functor

$$h: M \longmapsto Der(A, M)$$

$$A\text{-mod} \longrightarrow Set \qquad (*)$$

is corep^d by Ω_A $\left(h \cong \underset{A\text{-mod}}{Hom}(\Omega_A, -)\right)$

Cotangent Complex: $(A \in Alg_R)$

$$\mathbb{L}_A = R \underset{Bar(A)}{\otimes} \Omega_{Bar(A)} \in sCAlg_R \leftrightarrow cdga^{\leq 0}$$

Where

$$Bar(A) = \cdots A \times A \times A \rightrightarrows A \times A \rightarrow A$$

Do we have something like $(*)$? Recall

$$\begin{tikzcd} Der(A, M) \arrow[r, hook] & Hom(A, A \oplus M) \arrow[d] \\ & Hom(A, A) \end{tikzcd}$$

So in $cdga^{\leq 0}$ define Der as the fibre

$$\begin{tikzcd} Der(A, M) \arrow[r] & Map(A, A \oplus M) \arrow[d] \\ & Map(A, A) \end{tikzcd}$$

Then

Prop.

$$Der(A, M) \underset{\text{homotopic}}{\simeq} Map(\mathbb{L}_A, M)$$

Rmk: $\pi_0(\mathbb{L}_A) \simeq \Omega^1_{\pi_0(A)}$, "Higher homotopies measure obstruction to Smoothness".

Deformations about a point:

Let X be a scheme over k, $x \in X(k)$ ($x: \operatorname{Spec} k \to X$). Assume we want to deform X around x: "Thickenings" to spaces over $k \oplus k[i]$ ← i^{th}-order thickening (trivial square zero extension) Note: $k \oplus k[0] \simeq k[\varepsilon]/(\varepsilon)^2$

The space of such thickenings to a $k \oplus k[i]$ family is

$$\operatorname{Ext}^i_k(x^* \mathbb{L}_X, k)$$

Note.

$$\operatorname{Ext}^0_k(x^* \mathbb{L}_X, k) = \operatorname{Hom}_{\operatorname{Spec} k/\mathrm{Sch}}\left(\operatorname{Spec} k[\varepsilon_0], (X, x)\right)$$

but there is no such "std. interp. for higher i until we pass to derived schemes:

Prop:

Derived i^{th} order infinitesimal disk

$$\operatorname{Ext}^i_k(x^* \mathbb{L}_X, k) \cong \operatorname{Hom}_{\mathrm{Ho}(\operatorname{Spec} k/\mathrm{dSch})}\left(\operatorname{Spec} k \oplus k[i], (X, x)\right)$$