Recap on Yangians

$Y(\mathfrak{gl}_n)$

Motivation

1 in i,jth position

$\mathfrak{gl}_n$ is generated by n^2 (matrices) E_{ij}, $1 \le i,j \le n$:

$$[E_{ij}, E_{k\ell}] = \delta_{kj} E_{i\ell} - \delta_{i\ell} E_{kj}$$

Define

defining rep of E_{ij}: $V_a \cong \mathbb{C}^n$

$$E = \sum_{i,j} E_{ij}\, e_{ij} \in U(\mathfrak{gl}_n) \otimes \mathrm{End}(V_a)$$

Rmk:

$$g_s = \mathrm{Tr}\, E^s \in U(\mathfrak{gl}_n), \quad s = 1, 2, \ldots, n$$

generate the center of $U(\mathfrak{gl}_n)$

Now,

$$[(E^{r+1})_{ij}, (E^s)_{k\ell}] - [(E^r)_{ij}, (E^{s+1})_{k\ell}] \qquad (*)$$

$$= (E^r)_{kj}(E^s)_{i\ell} - (E^s)_{kj}(E^r)_{i\ell}$$

Def ($k=1$)

$Y(n) = Y(\mathfrak{gl}_n) :=$ Free unital assoc. Alg. gen by $t^{(0)}_{ij} = \delta_{ij}, t^{(1)}_{ij}, t^{(2)}_{ij}, \ldots \big/ I$

where I is the ideal gen. by $(*)$ w/

$$(E^r)_{ij} \rightsquigarrow (t^{(r)})_{ij}$$

Rmk

It is usually convenient to work over $Y[\![u^{-1}]\!] \otimes \mathrm{End}(V_a)^{\otimes m}$ for some m.

$$T(u) := \sum_{i,j=1}^{n} t_{ij}(u) \otimes e_{ij} \in Y[\![u^{-1}]\!] \otimes \mathrm{End}(V_a)$$

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots$$

$$T_a(u) := \sum_{i,j} t_{ij} \otimes 1^{a-1} \otimes e_{ij} \otimes 1^{m-a}$$

Then we can rewrite $(*)$ as the RTT relations

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v) \in Y[\![u^{-1}]\!] \otimes \mathrm{End}(V_a)^{\otimes 2}$$

with

$$\left.\begin{aligned} R(u) &:= 1 - \tau u^{-1} \\ \tau &:= \sum_{i,j} e_{ij} \otimes e_{ji} \end{aligned}\right. \quad \in \mathrm{End}(V_a)^{\otimes 2}$$

R satisfies the YB eqns in $\mathrm{End}(V_a)^{\otimes 3}$.

Prop

$$(**) \qquad \begin{aligned} \mathrm{eval}: Y(n) &\twoheadrightarrow U(\mathfrak{gl}_n) \\ t_{ij}(u) &\longmapsto \delta_{ij} + E_{ij} u^{-1} \end{aligned} \qquad \left(T(u) \longmapsto 1 + \tfrac{1}{u} E\right)$$

is an algebra epi ;

$$\begin{aligned} \iota: U(\mathfrak{gl}_n) &\hookrightarrow Y(n) \\ E_{ij} &\longmapsto t_{ij}^{(1)} \end{aligned}$$

is an embedding.

$(**) \Rightarrow$ Any $\mathfrak{gl}_n$-rep is a $Y(n)$-module.

Hopf Alg.

$Y(n)$ is a Hopf Algebra

Coproduct:

$$\Delta : T(u) \longmapsto T_{(1)}(u)\, T_{(2)}(u)$$

Antipode.

$$S : T(u) \longmapsto T(u)^{-1}$$

Counit

$$\varepsilon . \ T(u) \longmapsto 1$$

Reps (Modify to include automorphisms or anti-automorphisms)
Using the coproduct and evaluation maps, any $U(\mathfrak{gl}_n)^{\otimes N}$-module $\mathcal{H}$ becomes a $Y(n)$-module:

$$\rho(T(u)) = \rho_{(1)}(T(u)) \otimes \cdots \otimes \rho_{(N)}(T(u))$$

E.g.

$$\mathcal{H} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}. \qquad \mathfrak{gl}_n \curvearrowright V_{\lambda_i}$$

$$\rho_{(\ell)}(T(u)) = 1^{\otimes(\ell-1)} \otimes (\mathbb{1}_n + u^{-1}\rho(E)) \otimes 1^{(N-\ell-1)}$$

$Y(\mathfrak{sl}_n)$

1) $\exists$ an explicit construction in terms of $x_1, \ldots, x_n$ a basis of $\mathfrak{sl}_n$ and $J(x_1), \ldots, J(x_n)$ + commutation relations.

2) $Y(\mathfrak{sl}_n) \hookrightarrow Y(\mathfrak{gl}_n)$

(*) 3) $Y(\mathfrak{sl}_n)$ is a quotient of $Y(\mathfrak{gl}_n)$:

$$\left(Y(\mathfrak{sl}_n) = Y(\mathfrak{gl}_n) / (\mathrm{qdet}\, T(u) = 1) \right)$$

↑ Part of Thm.

Thm

1) $Y(\mathfrak{sl}_n)$ is a Hopf algebra by restriction of the Hopf algebra str of $Y(\mathfrak{gl}_n)$.

2) $Y(\mathfrak{gl}_n) \cong Z(n) \otimes Y(\mathfrak{sl}_n)$

Cor

$$Y(\mathfrak{sl}_n) \cong Y(\mathfrak{gl}_n) / (\operatorname{qdet} T(u) = 1)$$

As $\operatorname{qdet} T(u)$ generates the center.

(**Rmk**

$\exists$ an evaluation map

$$ev : Y(\mathfrak{sl}_n) \twoheadrightarrow U(\mathfrak{sl}_n)$$

s.t

$$\begin{tikzcd} U(\mathfrak{sl}_n) \arrow[r, hook] \arrow[rr, bend right, "id"'] & Y(\mathfrak{sl}_n) \arrow[r, two heads] & U(\mathfrak{sl}_n) \end{tikzcd}$$
)

Bethe Subalgebras

Fix $C \in \operatorname{End}(\mathbb{C}^n)$, then for $k = 1, \ldots, n$ define

$$B_k(u, C) = \operatorname{Tr}_{V^{\otimes n}}[A_n T_1(u) T_2(u-1) \cdots T_k(u-k+1) C_{k+1} \cdots C_n] \in Y(n)[[u^{-1}]]$$

where

$$A_n = \frac{1}{n!} \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \sigma \in \mathbb{C}[S_n] \curvearrowright \operatorname{End}(\mathbb{C}^n)^{\otimes n}$$

Thm

The coefficients of $B_k(u, C)$ generate a commutative subalg. (Max'l if C has simple spectrum,

(Coeff's are independent if C has simple spectrum).

Rmks

- Choice of C $\rightsquigarrow$ Choice of ∂-cond. on N-spin chain

  $C = \mathbb{1} \longmapsto$ Periodic ∂-cond.

- $B_n(u, \mathbb{1}) = q\det T(u) \implies$ coeff's generate $Z(n)$

Example: XXX Spin Chain : $Y(\mathfrak{gl}_2)$

$$\mathcal{H} = V_{1/2}^{\otimes N}$$

$$\rho(T(u)) = \rho_1 \left[ev(T(u)^T) \right] \cdots \rho_N \left[ev(T(u)^T) \right]$$

$$= L_1(u) \cdots L_N(u)$$

With

$$L(u) = \mathbb{1}_{V_a} + \frac{\tau}{u} \quad ; \quad \tau = \dot{E}^T = \sum_{ij} E_{ij} \otimes e_{ji}$$

$$\left(\begin{array}{c} \tau : V_{1/2} \otimes V_a \longrightarrow V_{1/2} \otimes V_a \\ x \otimes y \longmapsto y \otimes x \end{array} \right)$$

$$B_1(u, \mathbb{1}) = \mathrm{Tr}_{V_a}\left[T(u) \right]$$

$$\downarrow$$

$$F(u) = \mathrm{Tr}_{V_a}\left[L_1 \cdots L_N \right]$$

Counting

Consider evaluation modules of the form

$$\mathcal{H} = V_1 \otimes \cdots \otimes V_N \,, \quad V_i \text{ a } GL_n\text{-irrep.}$$

Rmk

$\Delta : Z(u) \longrightarrow Z(u) \otimes Z(u)$

Via evaluation, $Z(n) \overset{ev}{\twoheadrightarrow} Z[U(\mathfrak{gl}_n)]$. V_i an irrep $\Rightarrow$

$Z(n)$ acts on $\mathcal{H}$ via scalar multiples of $\mathbb{1}_{V_1} \otimes \cdots \otimes \mathbb{1}_{V_N} = \mathbb{1}_{\mathcal{H}}$.

Classically: Look at $SU(n)$ = compact real form of GL_n / Center.

$$\text{Phase Space} = \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_n} \subset (\mathfrak{su}_n^*)^{\times N}$$

$$\mathcal{O}_{\xi_i} = \text{Coadjoint orbit} \subset \mathfrak{su}_n^*$$

$$\mathcal{O}_{\xi_i} \overset{\text{Quant.}}{\rightsquigarrow} V_i$$

Now

$$\dim \mathcal{O}_{\xi_i} = \dim G - \operatorname{rk} G \qquad (G = SU(n))$$

So we need

$$\#\,\text{Commuting Ham.} = \tfrac{1}{2} N (\dim G - \operatorname{rk} G) = \tfrac{1}{2} N n (n-1) .$$

Quantum Mech

There are $\operatorname{rk} G$ (commuting) Hamiltonians coming from the diagonal action of GL_n on $\mathcal{H}$; 1 is a scalar coming from the action of the center $\Rightarrow$

$$D = \tfrac{1}{2} N n (n-1) - (n-1)$$

leftover.

Now,

$$B_K(u,\mathbb{1}) = \mathrm{Tr}_{V_a^{\otimes n}}\left[A_n T_1(u)\cdots T_K(u-K+1)\cdot \mathbb{1}_a^{\otimes(n-K)}\right]$$

with

$$T_\ell(u) \longmapsto \mathbb{1}_a^{\otimes(n-\ell)} \otimes \left[\left(\mathbb{1}_1 + \frac{P_1(E)}{u}\right)\otimes\cdots\otimes\left(\mathbb{1}_N + \frac{P_N(E)}{u}\right)\right]\otimes \mathbb{1}_a^{(n-\ell+1)}$$

So

$$B_K(u,\mathbb{1}) \longmapsto \mathrm{Const}_1 + u^{-1}\,\mathrm{Const}_2\cdot\left[\sum_{i=1}^{N}\mathrm{Tr}_{V_a}\left(P_i(E)\right)\right]+\cdots+(\cdots)\,u^{-NK}$$

but

$$\begin{aligned}\mathrm{Tr}_{V_a}\left[P_i(E)\right] &= \sum_{i,j} P_i(E_{ij})\cdot \mathrm{Tr}(e_{ji}) \\ &= P_i(E_{11}+\cdots+E_{nn}) \quad \leftarrow \mathrm{Tr}\,E \text{ (central)} \\ &= c\cdot \mathbb{1}_i\end{aligned}$$

Thus,

$$B_K \rightsquigarrow NK-1 \text{ Commuting Hams.}$$

B_n gives central elements; so

$$\begin{matrix}\text{Total \#}\\ \text{Bethe Hams.}\end{matrix} = \sum_{K=1}^{n-1}(NK-1) = \tfrac{1}{2}Nn(n-1)-(n-1) = D.$$

<u>Bethe Ansatz</u>: Back to $XXX_{1/2}$ Spin chain.

Here

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u)\end{pmatrix}$$

$$F(u) = \mathrm{Tr}_{V_a} T(u) = A(u) + D(u)$$

Finding the Spectrum of Bethe Subalg. $\rightsquigarrow$ Algebraic Bethe Ansatz.

Define

$$\Omega = \omega_1 \otimes \cdots \otimes \omega_N \quad , \quad \omega_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Note that:

$$L_i(u)\,\omega_i = \begin{pmatrix} u + \frac{i}{2} & * \\ 0 & u - \frac{i}{2} \end{pmatrix} \omega_i$$

$$\rightsquigarrow$$

$$T(u)\,\Omega = \begin{pmatrix} \overbrace{\left(u + \frac{i}{2}\right)^N}^{\alpha(u)} & * \\ 0 & \overbrace{\left(u - \frac{i}{2}\right)^N}^{\beta(u)} \end{pmatrix}$$

So

$$C(u)\,\Omega = 0$$
$$A(u)\,\Omega = \alpha(u)\,\Omega \quad \Rightarrow \quad \Omega \text{ Eigenvector for } A + D = F$$
$$D(u)\,\Omega = \delta(u)\,\Omega$$

To build other eigenvectors use "Raising" ops:

$$\phi(u_1, \ldots, u_\ell) = B(u_1) \cdots B(u_\ell)\,\Omega$$

ϕ an Eigenvector of $F = A + D$ $\rightsquigarrow$ A.B.A eqns.

$$\left(\frac{u_j + i/2}{u_j - i/2} \right)^N = \prod_{k \neq j}^{\ell} \frac{u_j - u_k + i}{u_j - u_k - i}$$