Stoked out About Stokes Groupoids (Orig: Get Stoked About Stokes Groupoids)

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Reference: M. Gualtieri, Songhao Li, Brent Pym 1305.7288

Motivation and Overview

Riemann - Hilbert Correspondence: X a (Complex) manifold

(holomorphic) Flat Connections | Parallel Transports

$$\{\mathcal{E},\ \nabla : \mathcal{E} \to \Omega^1_X(\mathcal{E})\} \underset{\text{DIFF.}}{\overset{\text{integrate}}{\rightleftarrows}} \{P_\gamma : \mathcal{E}_{\gamma(0)} \longrightarrow \mathcal{E}_{\gamma(1)},\ \gamma \text{ a path}\}$$

Fancier Language:

Flatness $\Leftrightarrow \nabla_{[\xi,\eta]} = [\nabla_\xi, \nabla_\eta]$

$\Rightarrow$ Rep$^{\text{n}}$ of Tangent "Algebroid"

$$\mathcal{T}_X \longrightarrow \mathrm{Der}(\mathcal{E})$$

Rep$^{\text{n}}$ of Fundamental Groupoid

$$\Pi_{\leq 1}(X) \longrightarrow \mathrm{Aut}(\mathcal{E})$$

So

$$\mathrm{Rep}(\mathcal{T}_X) \simeq \mathrm{Rep}(\Pi_{\leq 1}(X))$$

A Fancier Problem: Let us raise our pinkies high (Sip tea and raise our pinkies high)

- X a smooth complex curve
- D an effective divisor on X: $D = \sum_{i=1}^{n} \nu(p_i)\, p_i$, $\nu(p_i) \in \mathbb{Z}_{>0}$, $p_i \in X$.

Want to study Connections on X with Singularities bounded by D:

z a local coord around $p_i \in D$, $(\nabla, \mathcal{E})$ Flat bundle w/ local Frame:

$$\nabla = d + A(z)\, z^{-k}\, dz \quad , \quad A: \text{Holomorphic Matrix-Valued Function}$$

At worst Sing. of order k,

"bounded above by k."

Naive RH Correspondence:

$$\mathrm{Rep}(\mathcal{T}_{X\setminus D}) \simeq \mathrm{Rep}(\Pi_{\leq 1}(X, D))$$

Contains Connections w/ essential Sing on D.

Lose all local data around Sing.

Do not rewrite, reuse previous RH Statement

Appropriate Refinement:

$$\mathrm{Rep}(\mathcal{T}_X(-D)) \simeq \mathrm{Rep}(\Pi_{\leq 1}(X, D))$$

Sheaf of v.f. w/ Zeros bounded below by D

(locally $\langle z^k \frac{\partial}{\partial z} \rangle_{\mathcal{O}_{U \subset X}}$)

$\Rightarrow$ $\nabla_\xi : \mathcal{E} \to \mathcal{E}$ non-singular if $\xi \in \mathcal{T}_X(-D)$ and ∇ has Sing bounded by D

$$\underbrace{\Pi_{\leq 1}(X, D)|_{X \setminus D}}_{\parallel}$$

$$= \Pi_{\leq 1}(X \setminus D) \cup_\varphi \coprod_{p \in D} \mathrm{Sto}_{\nu(p)}|_{\mathbb{D} \ni p}$$

"Preserves local data at D."

<u>Claim</u>: By appropriate pullbacks to the (Lie Groupoid) $\Pi_{\leq 1}(X, D)$ we can take Fundamental Solutions / Parallel Transports of a diagonal Connection <u>Formally</u> equiv to another (non-diag. Conn.) to <u>actual</u> Sol'ns.

If $(\mathcal{E}_1, \nabla_1)$, $(\mathcal{E}_2, \nabla_2)$ are <u>Formally</u> equivalent, then by Pullbacks to $\Pi_{\leq 1}(X, D)$ we can determine P_2 from P_1: The Formal Sol$^{\underline{n}}$ $\hat{P}_2$ <u>Converges</u>.

(Holomorphic) Lie Groupoids : Groupoids whose arrows and objects are Complex mans.

Def

A Groupoid $G \overset{s}{\underset{t}{\rightrightarrows}} X$ is a hol. Lie Groupoid iF

1) G (arrows), X (objects) are $\mathbb{C}$-maniFolds [G possibly non-HausdorFF]

2) $s, t : G \longrightarrow X$ (source/target) are hol. Submersions

3) $m : G \,{}_s\!\times_t G \longrightarrow G$ is holomorphic

4) $id : X \hookrightarrow G$ (embedding oF identity arrows) is a Closed embedding

Ex:

1) $\bullet$

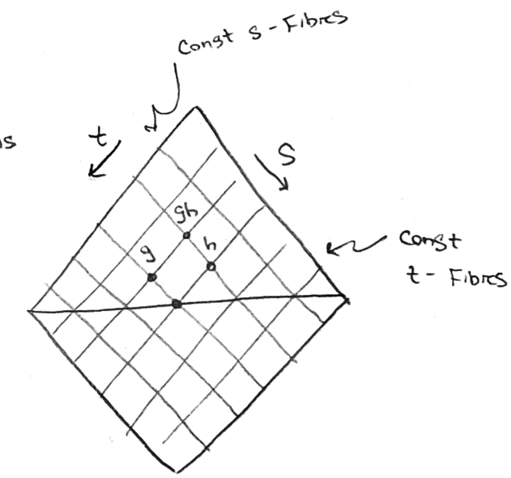
2) X a $\mathbb{C}$-man.

$$\text{Pair}(X) = X \times X \overset{s = p_1}{\underset{s = p_2}{\rightrightarrows}} X$$

Projections onto 1st and 2nd coords

$$(x, y) \cdot (y, z) := (x, z)$$

$$id = \Delta : X \hookrightarrow X \times X$$
$$x \longmapsto (x, x)$$



3) Gauge Groupoid: $\mathcal{E}$ a locally Free SheaF (Vector bun.), $\mathcal{E}_p$ = Fibre over p

$$\text{Aut}(\mathcal{E}) = \{ \mathcal{E}_p \xrightarrow{\sim} \mathcal{E}_q \ \ \mathbb{C}\text{-linear iso's For } p, q \in X \}$$

4) $\Pi_{\leq 1}(X) = \{ [\gamma] : \gamma \text{ a path on } X \}$, $s(\gamma) = \gamma(0)$, $t(\gamma) = \gamma(1)$.

Note: S-Fibres $s^{-1}(x_0)$ are the usual construction oF the universal Cover oF X using paths based at X.

$(s^{-1}(x_0) \cong \tilde{X})$.

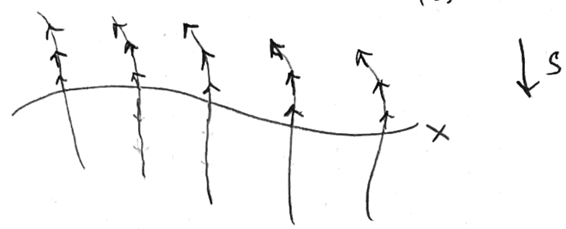
Lie Algebroids

Def

$$\mathrm{Lie}(G) := \underbrace{N_G X}_{\text{image of } id:\, X \hookrightarrow G} \overset{\text{canonical splitting using } s_*}{\cong} \mathrm{Ker}(s_*)$$

is a vector bundle over X, equipped w/ :

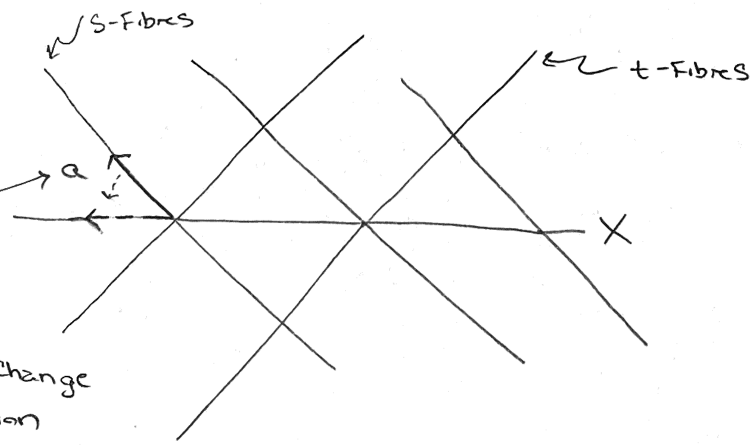
- $[\cdot,\cdot]$: Extend sections of normal bundle to Right invar V.F. on G (under groupoid action on itself) tangent to s fibres (use isotropy Groups)



Section of normal bundle $\updownarrow$ Element of $\mathrm{Lie}(G)$

- Anchor map $a : \mathrm{Lie}(G) \longrightarrow T_X$ given by $t_* = dt|_{\mathrm{Ker}(s_*)} : \mathrm{Ker}(s_*) \to T_X$.

Ex: • $\mathrm{Lie}(\mathrm{Pair}(X)) \cong T_X$ via a.



Infinitesimal change in t-direction Projected to X.

- $\mathrm{Lie}(\Pi_{\leq 1}(X)) \cong T_X$ (Same local picture)

See Version 1 For rest of Lecture (Contains more detail than Should be Said)
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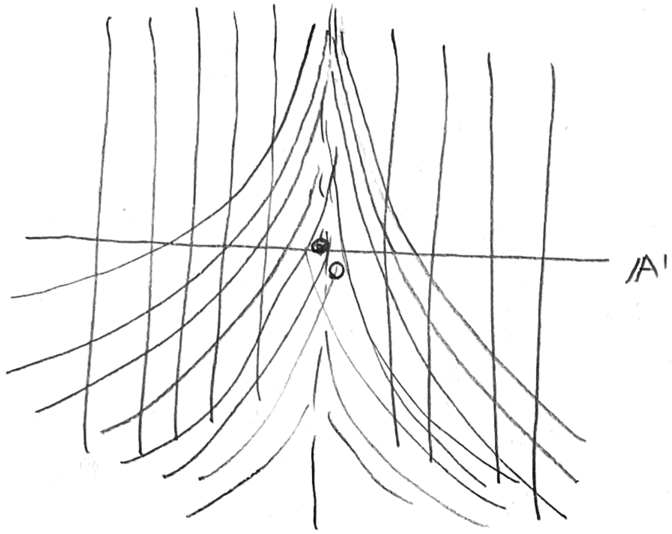
Ex: "Twisting" Groupoids along a Divisor.

- $\mathrm{Lie}(\mathrm{Pair}(X,D)) \simeq \mathrm{Lie}(\Pi_{\leq 1}(X,D)) \simeq \mathcal{T}_X(-D) \xrightarrow{a} \mathcal{T}_X$

$\mathrm{Sto}_1 := \Pi_{\leq 1}(\mathbb{A}^1, 1\cdot 0) \quad (\cong \mathrm{Pair}(\mathbb{A}^1, 1\cdot 0))$.



$\downarrow s$ Fibres

Curves: t-Fibres

Isotropy Grp @ p $= s^{-1}(pt) \cap t^{-1}(pt) = \begin{cases} * \cong \pi_1(\mathbb{A}^1, pt.) & \text{if } pt \neq 0 \\ \underset{\underset{\mathbb{C}.}{\cong}}{\mathbb{G}_a} \cong T_{pt}\mathbb{A}^1 & \text{if } pt = 0 \end{cases}$

Integrations

Def

An integration of an algebroid $(A, [\cdot,\cdot], a: A \to \mathcal{T}_X)$ is a pair (G, ϕ) with G a groupoid and $\phi: \mathrm{Lie}(G) \xrightarrow{\sim} A$.

Rmk / Defⁿ of $\Pi_{\leq 1}(X,D)$

- The Set of integrations, Forms a Category

- When $A = \mathcal{T}_X(-D)$ $\quad \mathrm{Pair}(X,D)$ is Final in this Category ⇜ Fibres $s^{-1}(pt.) \cong X$ For pt. generic

$\Pi_{\leq 1}(X,D)$ is initial ⇜ Source Simply Connected Fibres ⇒ unique up to iso

- $Pair(X,D)$ and $\Pi_{\leq 1}(X,D)$ can be constructed via "blowups" on $Pair(X)$ and $\Pi_{\leq 1}(X)$ or alternatively via glueing in copies of $Sto_K := \Pi_{\leq 1}(\mathbb{A}^1, K\cdot 0) = Pair(\mathbb{A}^1, K\cdot 0)$.

<u>Thm</u>

Let X be a complex curve, $D \subseteq X$ a divisor $D = \sum_{p\in X} \nu(p) D$; $U = X \setminus D$

1) $\Pi_{\leq 1}(X,D)|_{X\setminus D} = \Pi_{\leq 1}(X,D) \setminus (s^{-1}(U) \cup t^{-1}(U)) \cong \Pi_{\leq 1}(U)$

2)

$$\Pi_{\leq 1}(X,D) \simeq \Pi_1(U) \cup \coprod_{p\in D} Sto_{\nu(p)}|_{\mathbb{D}}$$

Gluing map different for $\mathbb{P}^1$, $D = K\cdot pt$.

If U is non-contractible, then the resulting space is Hausdorff; (gluing is via the map $\varphi : \Pi_{\leq 1}(U \cap V) \longrightarrow \coprod_{p\in D} Sto_{\nu(p)}|_{\mathbb{D}^*}$.

<u>Extension of Solutions over Singularities</u>

<u>Thm</u>

$$Rep(\mathcal{T}_X(-D)) \cong Rep(\Pi_{\leq 1}(X,D))$$

G-equivariant

Objects: $(\mathcal{E}, \nabla)$, ∇ has sing. bounded by D.

$\updownarrow$

$$\mathcal{T}_X(-D) \longrightarrow Der(\mathcal{E})$$

$$\xi \longmapsto \nabla_\xi : \mathcal{E} \to \mathcal{E}$$

$$(\nabla : \mathcal{E} \to (\mathcal{T}_X(-D))^\vee \otimes \mathcal{E}).$$

Obj: $(\mathcal{E}, P)$, P :

$$\begin{array}{ccc} s^*\mathcal{E} & \xrightarrow{\sim} & t^*\mathcal{E} \\ \downarrow & & \downarrow \\ \Pi_{\leq 1}(X,D) & \xrightarrow{id} & \Pi_{\leq 1}(X,D) \end{array}$$

$\updownarrow$

$$P : \Pi_{\leq 1}(X,D) \longrightarrow Aut(\mathcal{E})$$

$$\gamma \longmapsto P|_\gamma : \mathcal{E}_{s(\gamma)} \to \mathcal{E}_{t(\gamma)}$$

"Parallel transport"

$s(\gamma) = \gamma(0)$
$t(\gamma) = \gamma(1)$
if γ is a path, i.e. in $\Pi_{\leq 1}(X\setminus D)$

Constructing $\mathcal{P}$ in practice: $\mathcal{P}$ From Crap

Find Fundamental Solutions: $\{S_i \in \mathcal{E}\}_{i=1}^{r = \operatorname{Rank} \mathcal{E}}$

$$\nabla S_i = 0$$

"Stokes Phenomena"
S_i have varying asymptotics toward the divisor

$\Rightarrow$ Framing

$$\psi : (\mathcal{O}_X^r, d) \xrightarrow{\sim} (\mathcal{E}, \nabla) \qquad (*)$$

$$(f_1, \dots, f_r) \longmapsto f_1 S_1 + \cdots + f_n S_n$$

Unless $\mathcal{E}$ is trivial and D is trivial

$$\left\{ \begin{array}{l} \text{(a) } \psi \text{ is multivalued} \longrightarrow \text{Monodromy Matrices} \\ \text{(b) } \psi \text{ is singular along } D \text{ (w/ Stokes-type asymptotics)} \rightsquigarrow \text{Stokes Matrices} \end{array} \right\}$$

(me "Speculating")

(a) $\longleftrightarrow$ $\mathcal{P}_\gamma$ For $\gamma \in \Pi_{\leq 1}(X \setminus D) = \Pi_{\leq 1}(X, D)\big|_U \cong G_a$

(b) $\longleftrightarrow$ $\mathcal{P}_\gamma$ For $\gamma \in \Pi_{\leq 1}(X, D)\big|_D \supseteq \coprod_{p \in D} (T_p^* X)^{\otimes (\gamma(p) - 1)}$

Isotropy Groups

$\operatorname{iso}_p(\nabla) \in (T_p X)^{\otimes K} \otimes \operatorname{End}(\mathcal{E}_p)$

K-matrices attached to K-directions

Proposition

For any Fundamental Solution ψ as in $(*)$, the expression

$$\delta\psi = t^* \psi \circ (s^* \psi)^{-1}$$

extends holomorphically to $\Pi_{\leq 1}(X, D)$ and coincides w/ $\mathcal{P}$. Multi-valuedness is removed by requiring $\delta\psi\big|_X = 1$ (over identity bisection).

Ex: (only if there is enough time).

Rank 1 Repⁿ For $\Upsilon_{A'}(-\kappa\cdot p)$: $(\mathcal{O}_{A'}, \nabla)$ w/

$$\nabla = d + a z^{-\kappa} dz$$

We have multi-valued Fundamental Solⁿs:

$$\psi_1 = z^{-a} \quad , \kappa = 1$$
$$\psi_\kappa = \exp\left\{ \frac{a z^{-(\kappa-1)}}{\kappa - 1} \right\} , \quad \kappa > 1$$

Which Give ((z,u) coords on Sto_κ)

$$P_1 \big|_{(z,u)} = e^{-au}$$

$$P_2 \big|_{(z,u)} = e^{-a S_\kappa} , \quad S_\kappa = \frac{1 - e^{-u(\kappa-1) z^{\kappa-1}}}{(\kappa-1) z^{\kappa-1}}$$

Summation of Divergent Series

Motivation

Fundamental Solutions For Diagonal Connections are easy, Want g a hol. gauge transf. ($\in \mathrm{Aut}\,\mathrm{bun}(\mathcal{E})$). S.t.

Semi-simple (diagonalizable) Matrix

$$\nabla = d + \left(\frac{T_\kappa}{z^\kappa} + \cdots + \frac{T_1}{z} \right) dz \xrightarrow{g^*} g^*\nabla = d + \left(\frac{T_\kappa^{\,diag}}{z^\kappa} + \cdots + \frac{T_1^{\,diag}}{z} \right) dz$$

(ignore hol. stuff)

Can Find order by order : Caveat: Most of the time g is a <u>Formal</u> power series, i.e. has zero radius of convergence.

Theorem / Observation

Let $\hat{g}$ be a Formal iso between $\mathcal{T}_X(-D)$ reps ($((\mathcal{E}_1,\nabla_1)$ & $(\mathcal{E}_2,\nabla_2))$)

$$\begin{tikzcd} (\hat{\mathcal{E}}_1, \hat{\nabla}_1) \arrow[r, "g"] \arrow[d] & (\hat{\mathcal{E}}_2, \hat{\nabla}_2) \arrow[d] \\ \hat{X} \arrow[r, "id"] & \hat{X} \end{tikzcd}$$

$\hat{X}$ = Formal nbhd of D (Formal Completion of D)

and $P_i : s^*\mathcal{E}_1 \longrightarrow t^*\mathcal{E}_2$ the corresponding $\Pi_{\leq 1}(X,D)$ reps define $\hat{L}$ via:

Formal Parallel transport

$$\begin{tikzcd} s^*\hat{\mathcal{E}}_1 \arrow[r, "\hat{P}_1 = P|_{\hat{\mathcal{E}}_1}"] \arrow[d, "s^*\hat{g}"'] \arrow[dr, phantom, "\hat{L} \atop \shortparallel"] & t^*\hat{\mathcal{E}}_1 \arrow[d, "t^*\hat{g}"] \\ s^*\hat{\mathcal{E}}_2 \arrow[r, "t^*\hat{g} \circ \hat{P}_1 \circ (s^*\hat{g})"'] & t^*\hat{\mathcal{E}}_2 \end{tikzcd}$$

Then $\hat{L} = \hat{P}|_2$, i.e. $\hat{L}$ extends to a holomorphic/convergent parallel transport op. P.

(PF: Trivial, $\hat{P}_2 = P|_{\hat{\mathcal{E}}_2}$ is the unique operator that fits into the bottom arrow above assuming Formal $\mathcal{T}_X(-D)$ reps are in 1:1 correspondence with Formal $\Pi_{\leq 1}(X,D)$ - reps.)

Ex: Rep$^{\underline{n}}$s of $\mathcal{T}_{\mathbb{A}^1}(-2\cdot 0)$

Let

$$\nabla_1 = d + \begin{pmatrix} -1 & \\ & 0 \end{pmatrix} z^{-2}dz, \qquad \mathcal{E}_2 = d + \begin{pmatrix} -1 & z \\ & 0 \end{pmatrix} z^{-2}dz$$

then

$$g \circ \nabla_1 = \nabla_2 \circ g \iff z^2 g' = g - z$$

Actual Sol$^{\underline{n}}$ is C^∞ but not hol. at $z=0$.

$\exists$ a Formal Solution:

$$\hat{g}(z) = \sum_{n=0}^{\infty} n!\, z^{n+1}$$

P_1 is a parallel transport op on $\mathrm{Sto}_z = \Pi_{\leq 1}(\mathbb{A}', 2\cdot 0)$ $\Leftarrow$ (use $\mathrm{Pair}(\mathbb{A}', 2\cdot 0)$ instead)

$$P_1 = \begin{pmatrix} e^{u(1+zu)^{-1}} & \\ & 1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & t^* g \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{u(1+zu)^{-1}} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -s^* g \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{u(1+zu)^{-1}} & s^* g - e^{u(1+zu)^{-1}} s^* g \\ & 1 \end{pmatrix}$$

where

$$s(z, \mu) = z \qquad , \quad \mu = u(1+zu)^{-1}$$
$$t(z, \mu) = z(1 - z\mu)$$

Then we find

$$s^* \hat{g} - e^{\mu} s^* \hat{g} = - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{z^{i+1} \mu^{i+j+1}}{(i+1)(i+2)\cdots(i+j+1)}$$

a convergent power series for

$$g(z; \mu) = e^{\frac{z\mu - 1}{z}} \left(\mathrm{Ei}\left(\frac{1 - z\mu}{z}\right) - \mathrm{Ei}\left(\frac{1}{z}\right) \right)$$