Fourier-Mukai: A Perspective From the Village idiot

Recall: Undergraduate Career: The Fourier Transform / Pontryagin Duality

Setup: G a locally compact abelian group (e.g. $\mathbb{R}, \mathbb{T}, \mathbb{Z}$)

$G^\wedge := \{\text{Continuous group homs } G \longrightarrow \mathbb{T}\}$ also LCA

(e.g. $\mathbb{T}^\vee \cong \mathbb{Z}$, $\mathbb{Z}^\vee \cong \mathbb{T}$, $\mathbb{R}^\vee \cong \mathbb{R}$) ; Pontryagin Duality: $G^{\wedge\wedge} \cong G$

- LCA $\Rightarrow$ $\exists$ a Haar measure (unique up to scaling)

- Fourier Transform:

$$\begin{tikzcd} & G \times G^\vee \arrow[dl, "q"'] \arrow[dr, "p"] & \\ G & & G^\vee \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & \mathrm{Fun}(G \times G^\vee) \arrow[dr, "p_* = \int_G (-)\, d\mu"] & \\ \mathrm{Fun}(G) \arrow[ur, "q^*"] & & G^\vee \end{tikzcd}$$

$$\mathcal{F}(f)(\chi) = \int_G f(x)\, \overline{\chi(x)}\, d\mu(x) \quad , \text{ i.e.}$$

$$\mathcal{F} : f \longmapsto p_*[q^* f \cdot X]$$

($\mathbb{R}^n$: $\chi_p(x) = e^{i\langle p, x\rangle}$ for some $p \in (\mathbb{R}^n)^*$).

where
$$X : G \times G^\vee \longrightarrow \mathbb{C}$$
$$(x, \chi) \longmapsto \overline{\chi(x)}.$$

Properties

- $\mathcal{F} : L^2(G) \rightleftarrows L^2(G^\wedge) : \mathcal{F}^+$ (left adjoint) $\longrightarrow$ $\langle \mathcal{F} f, g \rangle = \langle f, \mathcal{F}^+ g\rangle$.

- $\mathcal{F}$ is unitary (Parseval's Thm.): $\mathcal{F}^+\mathcal{F} = \mathcal{F}\mathcal{F}^+ = \mathbb{1}$ (both left & right adjoints)

$\Rightarrow \langle \mathcal{F} f, \mathcal{F} g\rangle_G = \langle f, g\rangle_{G^\vee}$ (follows from $\mathcal{F}^+\mathcal{F} = \mathbb{1}$)

Recall: Convolution: $f * g : x \longmapsto \int_G f(x-y)\, g(y)\, d\mu(y)$

- $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$

- $\mathcal{F}(f \cdot g) = \mathcal{F}(f) * \mathcal{F}(g)$

- δ-Functions: $\mathcal{F}(\delta(x)) : \chi \longmapsto \overline{\chi(x)}$ ↜ Supp $\mathcal{F}(\delta(x)) = G^{\vee}$

  $\mathcal{F}(x \longmapsto \chi(x)) = \delta(\chi)$ ↜ Supp $\mathcal{F}$ = point @ χ

Fourier - Mukai: A "Categorification" of the above for Sheaves

Setup: X, Y Schemes ; $\mathcal{E} \in Sh(X \times Y)$

$$\begin{tikzcd} & X \times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y \end{tikzcd}$$

Define

$$\Phi_{\mathcal{E}} : Sh(X) \longrightarrow Sh(Y)$$
$$\mathcal{F} \longmapsto p_*(q^* \mathcal{F} \otimes \mathcal{E})$$

Want $\Phi_{\mathcal{E}}$ to behave like Fourier - Transform, in particular, want Left / Right Adjoints $\Rightarrow$ pass to derived Categories (need Verdier-duality).

Basic Derived Stuff : Squiggly arrows

Derived Categories: Where Cool Kids do homological algebra

Setup: A an abelian Category (w/ enough "injectives" & "projectives").

X a Scheme:

$$D^b(X) = \text{"bounded derived Category of Coherent Sheaves on } X\text{"}$$

$$\text{"="}\ Ch[Coh(X)][\text{quasi-iso}]^{-1}$$

Then for X, Y Schemes and $\mathcal{E}^\bullet \in D^b(X)$:

$$\Phi_{\mathcal{E}} : D^b(X) \longrightarrow D^b(Y)$$
$$\mathcal{F}^\bullet \longmapsto Rp_*(Lq^*\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{E}^\bullet)$$

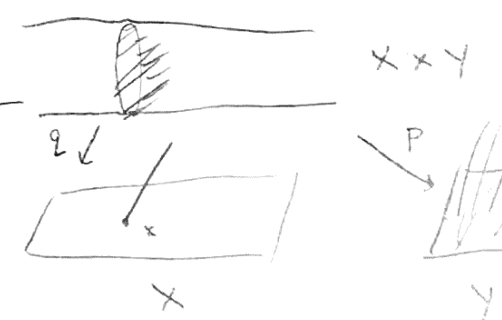
Restrict to
- X, Y smooth projective Varieties $/k$
- $\mathcal{E}^\bullet$ Complex of locally-Free Sheaves (e.g. Vector bundles)

Then
- Lq^* is just q^* on Complexes (Via Smoothness)
- $\overset{L}{\otimes}$ is just $\otimes$ on complexes (Via locally Free)

Ex: • $x \in X$ a Closed point; $k(x)$ the Skyscraper Sheaf, $\mathcal{E} \in Coh(X)$ locally Free

$$\Phi_{\mathcal{E}}(k(x)) = Rp_*(q^* k(x) \otimes \mathcal{E})$$
$$\simeq \mathcal{E}|_{\{x\}\times Y}$$

Sheaf on Y



• $id\ D^b(X) \longrightarrow D^b(X) \simeq \Phi_{\mathcal{O}_\Delta}$,

$\mathcal{O}_\Delta$ = Structure Sheaf of $\Delta \subset X \times X$:

Let $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$, then

$$\Phi_{\mathcal{O}_\Delta}(\mathcal{F}^\bullet) = P_*(q^*\mathcal{F}^\bullet \otimes \mathcal{O}_\Delta)$$
$$= P_*(q^*\mathcal{F}^\bullet \otimes \iota_*\mathcal{O}_X)$$

$$\cong p_*\left(\iota_*\left(\iota^* q^* \mathcal{E}^\bullet \otimes \mathcal{O}_X\right)\right) \quad \text{(projection Formula)}$$

$$\cong (p \circ \iota)_* (q \circ \iota)^* \mathcal{E}^\bullet \qquad (p \circ \iota = \mathrm{id} = q \circ \iota)$$

$$\cong \mathcal{E}^\bullet$$

- $f : X \longrightarrow Y$. $\Gamma_f = \operatorname{graph}(f) \subset X \times Y$. Then

$$f_* \cong \Phi_{\mathcal{O}_{\Gamma_f}} : D^b(X) \longrightarrow D^b(Y)$$

- $-[1] : \mathcal{E}^\bullet \longmapsto \mathcal{E}^\bullet[1] \cong \Phi_{\mathcal{O}_\Delta[1]}$

<u>Orlov</u>: Every Equivalence $D^b(X) \xrightarrow{\sim} D^b(Y)$ is (isomorphic to) a Fourier-Mukai Functor Φ_P for P unique up to isomorphism. X, Y smooth projective.

<u>Composition</u>. (Fubini's Thm.)

X, Y, Z smooth, projective; $\mathcal{E}^\bullet \in D^b(X \times Y)$; $\mathcal{F}^\bullet \in D^b(Y \times Z)$

$$D^b(X) \xrightarrow{\Phi_{\mathcal{E}}} D^b(Y) \xrightarrow{\Phi_{\mathcal{F}}} D^b(Z) \cong \Phi_R$$

where

$$R^\bullet := (\pi_{XZ})_* \left(\pi_{XY}^* \mathcal{E}^\bullet \otimes \pi_{YZ}^* \mathcal{F}^\bullet\right)$$

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "\pi_{XY}"'] \arrow[d, "\pi_{YZ}"] \arrow[dr, "\pi_{XZ}"] & \\
X \times Y & Y \times Z & X \times Z
\end{tikzcd}$$

Parseval's Thm:

Mukai: $\Phi_{\mathcal{E}}$ admits left & right adjoints

Rmk: Requires Verdier-duality => Derived category machinery is necessary

Prop. Let X, Y be smooth, proper alg. varieties; ω_Y the canonical bundle on Y. Suppose $\mathcal{E} \in Coh(X \times Y)$ is s.t.

- $\mathrm{Hom}_{D(X)}(\mathcal{E}_{x_1}, \mathcal{E}_{x_2}[i]) = \mathrm{Ext}^i(\mathcal{E}_{x_1}, \mathcal{E}_{x_2}) = 0$ $\forall i \in \mathbb{Z}$ whenever $x_1 \neq x_2$

  ($\mathrm{Ext}^i(\mathcal{E}_{x_1}, \mathcal{E}_{x_2}) = H^i(X; \mathcal{E}_{x_1}^* \otimes \mathcal{E}_{x_2})$)

- $\mathrm{Hom}_{Coh(X \times Y)}(\mathcal{E}, \mathcal{E}) \cong k$ (Automorphisms are constant)

Then

$$\Phi_{\mathcal{E}} : D(X) \longrightarrow D(Y) \text{ is an equivalence if } \dim(X) = \dim(Y)$$

and $\mathcal{E}_x \otimes \omega_Y \simeq \mathcal{E}_x \quad \forall x \in X.$

Rmk: The inverse to $\Phi_{\mathcal{E}}$ is given by $\Phi_{\mathcal{Q}}$ w/

$$\mathcal{Q} = \mathcal{E}^* \otimes \omega_X [\dim(X)]$$

Abelian Varieties

Def

An abelian variety is a projective connected algebraic group (over k).

Facts

(1) An AV is smooth and commutative

(2) If $k = \mathbb{C}$ then an AV is a complex Lie group $\cong \mathbb{C}^g/\Gamma$ (a complex torus)

For (2):

$$\begin{aligned} A &\longrightarrow H^0(A; \Omega^1)^* \\ a &\longmapsto \int_{P_{e,a}} : \omega \longmapsto \int_{P_{e,a}} \omega \end{aligned}$$

Descends to an iso

$$A \longmapsto H^0(A;\Omega^1)^* / H_1(A;\mathbb{Z})$$

$$\left(\int_\gamma \omega = 0 \iff [\gamma] \in H_1(A;\mathbb{Z}) \right)$$

But $H^0(A;\Omega^1) \cong \mathbb{C}^g$ For Some g ; as A is Compact $\iota : H_1(A;\mathbb{Z}) \hookrightarrow H^0(A;\Omega^1)$ Must be Full rank $\Rightarrow$ $A \cong \mathbb{C}^g/\Gamma$, $\Gamma \cong H_1(A;\mathbb{Z})$.

$$\text{Abelian Variety} \longleftrightarrow \text{Complex torus } \mathbb{C}^g/\Gamma$$

$+$ ample line bundle $\leftarrow$ Gives projective embedding.

<u>Dual Variety</u>.

$$\hat{A} := H^1(A;\mathcal{O}) / H^1(A;\mathbb{Z}) \longrightarrow \mathrm{Pic}(A) = H^1(A;\mathcal{O}^\times) \xrightarrow{c_1} H^2(A;\mathbb{Z})$$

$$= \mathrm{Pic}^0(A)$$

$$\cong \{ L \in \mathrm{Pic}(A) : t_a^* L \simeq L \ \forall a \in A \}$$

translation by a : $t_a : b \longmapsto b+a$

- t_a^* acts trivially on $H^1(A;\mathcal{O})$ $\Rightarrow$

- Translation invariant $\Rightarrow$ $\iota^* L \simeq L^*$ $(n^* L \simeq L^n)$

inversion

$\Rightarrow$ $\iota^* c_1(L) = -c_1(L)$ but ι^* acts as I on H^2 and $H^2(A;\mathbb{Z})$ is torsion-Free (it is a torus).

Rmk

$$\mathrm{Pic}^\circ(A) \cong \{ \text{Characters } \alpha \ \Gamma \longrightarrow \mathbb{T} \}$$

Indeed, take the trivial line bundle

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}^g & \xrightarrow[\text{by equivariant } \Gamma\text{-action}]{\text{Quotient}} & \mathbb{C} \times \mathbb{C}^g/\Gamma \\ \downarrow & & \downarrow \\ \mathbb{C}^g & & \mathbb{C}^g/\Gamma \end{array}$$

$\lambda \longmapsto$

$(v, a) \longmapsto (\alpha(\lambda) \cdot v, a + \lambda)$

Poincaré Line Bundle

$\exists!$ Line bundle $P \longrightarrow \hat{A} \times A$ s.t.

1) $P|_{\{\alpha\} \times A} \simeq L_\alpha$ (line bundle corresponding to $\alpha \in Hom(\Gamma, \mathbb{T})$)

2) $P|_{\hat{A} \times e} \simeq \mathcal{O}_{\hat{A}}$ (normalisation)

Poincaré - Bundle as the Fourier - Mukai Kernel

$$P \rightsquigarrow \Phi_P : D^b(A) \longrightarrow D^b(\hat{A})$$

OR

$$\Phi_{P^\vee} : D^b(\hat{A}) \longrightarrow D^b(A)$$

Apply Mukai's Thm:

inversion

$$D^b(A) \xrightarrow{\Phi_P} D^b(\hat{A}) \xrightarrow{\Phi_P} D^b(A) \cong \iota^* \circ [-\dim(A)]$$

Parseval:

$$Hom_{D(A)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[k]) \simeq Hom_{D(\hat{A})}(\Phi_P(\mathcal{F}^\bullet), \Phi(\mathcal{G}^\bullet)[k])$$

Convolution.

$$* : D^b(A) \times D^b(A) \longrightarrow D^b(A)$$

undecrived outer tensor

$$(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \longmapsto \underbrace{m}_{\text{Multiplication } A \otimes A \to A}{}_*(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$$

Prop:

$$\Phi_P(\mathcal{F}^\bullet * \mathcal{G}^\bullet) \simeq \Phi_P(\mathcal{F}^\bullet) \otimes \Phi_P(\mathcal{G}^\bullet)$$

$$\Phi_P(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) \simeq \Phi_P(\mathcal{F}^\bullet) * \Phi_P(\mathcal{G}^\bullet)[\dim(A)]$$

$SL_2\mathbb{Z}$ - Action

$\chi(L) = 1$

Let (A, L) be a principally polarized abelian Variety (e.g. $\mathrm{Jac}(C)$, w/ a point of C selected)

$\deg(L) = 1$

$$\varphi_L : A \xrightarrow{\sim} \hat{A}$$

$$a \longmapsto t_a^* L \otimes L^*$$

Now define

$$\Phi := \varphi_L^* \circ \Phi_P : D^b(A) \xrightarrow{\sim} D^b(A)$$

Prop (Mukai): Modulo - Shifts

$$SL_2\mathbb{Z} \text{ "}\longrightarrow\text{" } \mathrm{Aut}(D^b(A))$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longmapsto \Phi$$

$S^4 = 1$

$(T \circ S)^3 = 1$.

$$T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \longmapsto L \otimes (-)$$

Proof:

$$\Phi^4 \simeq [-2\dim(A)] \; ; \quad (L \otimes (-) \circ \Phi)^3 \simeq [-\dim(A)]$$