Higher Information

The untold topological secrets of measures and states

Tom Mainiero

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November 8, 2023

Related Work

- Baudot-Bennequin: *The Homological Nature of Entropy*;
- Vigneaux: *The structure of information from probability to homology*;
- Bennequin, Sergant-Perthuis, Vigneaux: *Extra-fine sheaves and interaction decompositions*;
- Baez-Fritz-Leinster: *Entropy as a Functor*;
- Hamilton/Leditsky: *Probing multipartite entanglement through persistent homology*;


PART I
THE BIG PICTURE
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What's the Big Idea?

Multipartite Measures $\longrightarrow$ "Spaces"

E.g.:

- (Purely Classical): Bipartite joint measures

  $$\widehat{\mu}\colon \Omega_{\mathsf{A}} \times \Omega_{\mathsf{B}} \longrightarrow \mathbb{R}_{\geq 0},$$

  Ω_i (finite) sets.

- (Purely Quantum): Bipartite pure states

  $$\psi \in \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}},$$

  $\mathcal{H}_{\mathsf{A}}$, $\mathcal{H}_{\mathsf{B}}$ Hilbert spaces.

- States assigned to causal diamonds on spacetime (local nets);

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Emergent (not a priori!) geometry/topology encodes correlations among various subsystems.

- (Weighted) Simplicial Complexes
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Emergent (not a priori!) geometry/topology encodes correlations among various subsystems.

- (Weighted) Simplicial Complexes
- (Co)simplicial objects in a category
- $\underbrace{\text{Pre-cosheaves of measures}}_{\text{"\textit{Measure Families}"}}$

What's the Big Idea? Practical Implications

Multipartite Measures ⟶ "Spaces"

- New topologically-inspired measures of shared information and entanglement;

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- New topologically-inspired measures of shared information and entanglement;
- Link invariants: $L \subset S^3$ a link with N-components;[1]

$$\mathcal{Z}_{\mathrm{CS}}[S^3 - L] \in \mathcal{Z}_{\mathrm{CS}}[\mathbb{T}]^{\otimes N} \rightsquigarrow \begin{cases} \text{Cohomology } H^\bullet \\ \text{Poincaré polynomial } \sum_i (\dim H^i) z^i \end{cases}$$

[1] Based on conversations with G. Moore. See work of Swingle and Balasubramanian, et. al.

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- Possible goal: new geometric proofs/categorifications of entropy inequalities improving on arguments using the Ryu-Takayanagi formula for holographic states.

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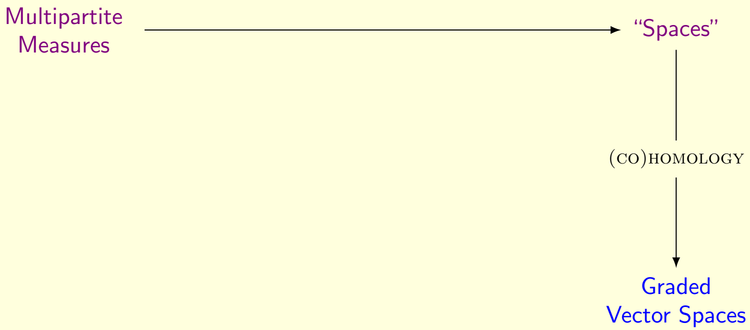
What's the Big Idea? Cohomology



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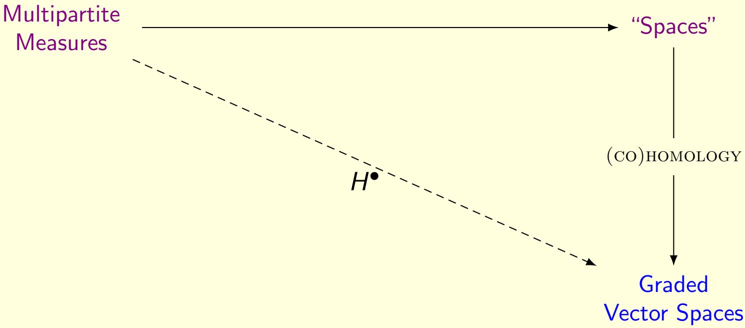

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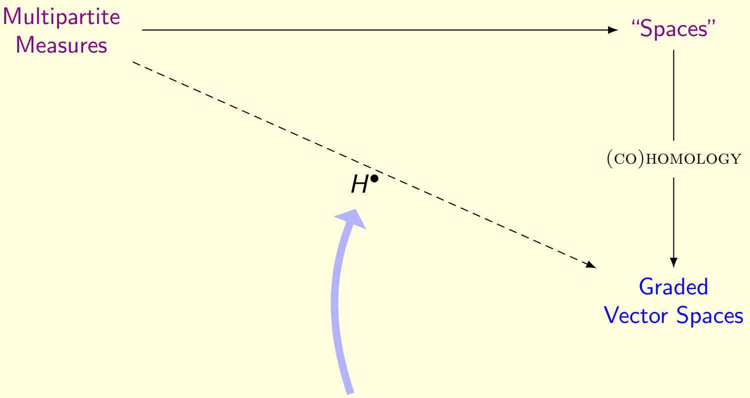

Explored in Detail in Homological Tools for the Quantum Mechanic (arXiv:1901.0211).

Cohomological Breadcrumbs

$$\underbrace{\mu}_{\text{Expectation Value}} : \overbrace{\mathrm{Rand}_{\mathbb{C}}(\Omega_{\mathsf{A}}) \otimes \mathrm{Rand}_{\mathbb{C}}(\Omega_{\mathsf{B}})}^{\mathrm{Rand}_{\mathbb{C}}(\Omega_{\mathsf{A}} \times \Omega_{\mathsf{B}})} \longrightarrow \mathbb{C}$$

Factorizability: $\mu = \mu_{\mathsf{A}} \otimes \mu_{\mathsf{B}}$

Descent of data to subsystems. All global data comes from gluing local data:

$$\mu\left(\sum_{ij} a_i \otimes b_j\right) = \frac{1}{\mu(1)} \sum_{ij} \mu(a_i \otimes 1)\mu(1 \otimes b_j).$$

Failure to Factorize

Obstruction to descent:

$$\mu(a \otimes b) \neq \frac{1}{\mu(1)} \mu(a \otimes 1)\mu(1 \otimes b)$$

for some $(a, b) \in \mathrm{Rand}_{\mathbb{C}}(\Omega_{\mathsf{A}}) \times \mathrm{Rand}_{\mathbb{C}}(\Omega_{\mathsf{B}})$.

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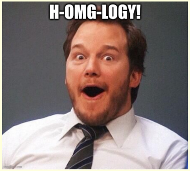
HOMOLOGICAL ALARM BELLS!



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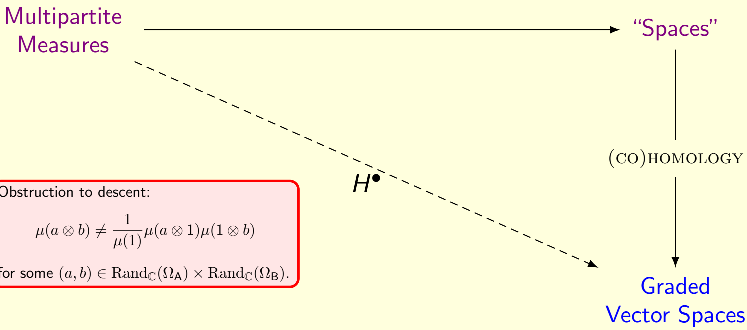

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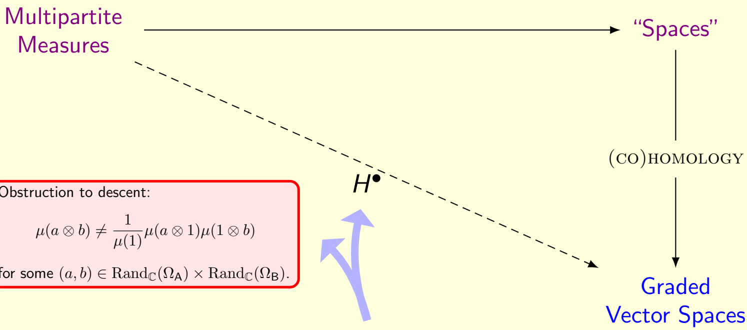

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$$H^0 \begin{bmatrix} \text{bipartite} \\ \text{measure} \\ \text{on } X \times Y \end{bmatrix} = \left\{ (x, y) \in \mathrm{Rand}_{\mathbb{C}}(X) \times \mathrm{Rand}_{\mathbb{C}}(Y) : \begin{smallmatrix} x \text{ and } y \\ \text{are} \\ \text{maximally correlated} \end{smallmatrix} \right\} / \mathbb{C}\langle(1,1)\rangle$$

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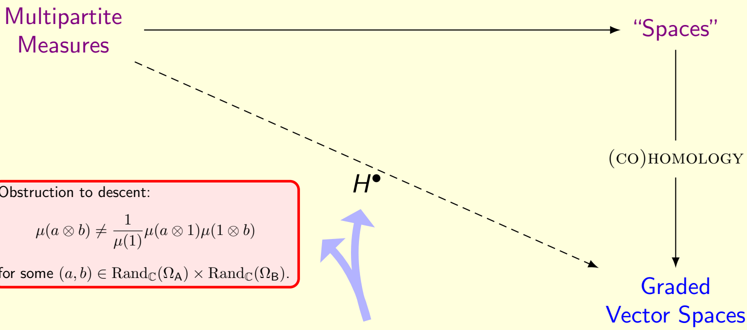

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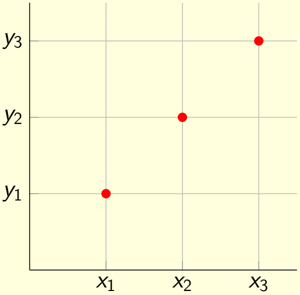
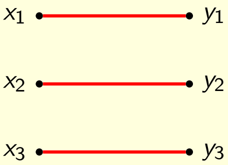
$$H^k \begin{bmatrix} N\text{-partite} \\ \text{measure} \end{bmatrix} = \left\{ \begin{matrix} \text{tuples of } (k+1)\text{-body random variables} \\ \text{exhibiting correlations} \end{matrix} \right\} \Big/ \left\{ \begin{matrix} \text{trivial} \\ \text{correlations} \end{matrix} \right\}, \ k \leq N - 2$$

Cohomology of a (Commutative) Bipartite Measure

Support of $\widehat{\mu}\colon X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu :=$



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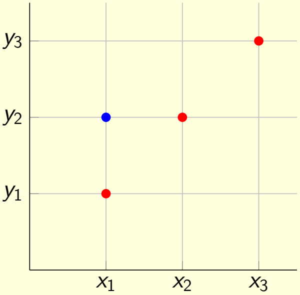

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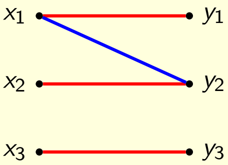

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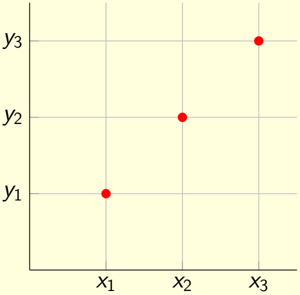

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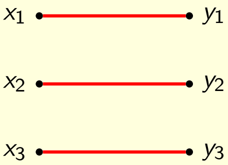

$$H^0(G_\mu; \mathbb{C}) \cong \mathbb{C}^{\#\text{ of connected components}} = \mathbb{C}^3$$

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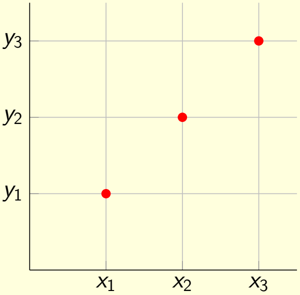

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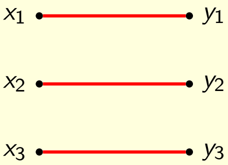

$$H^0(G_\mu; \mathbb{C}) = \mathbb{C}\langle (1_{\{x_1\}}, 1_{\{y_1\}}), (1_{\{x_2\}}, 1_{\{y_2\}}), (1_{\{x_3\}}, 1_{\{y_3\}}) \rangle$$

$1_S :=$ the indicator function on the subset S

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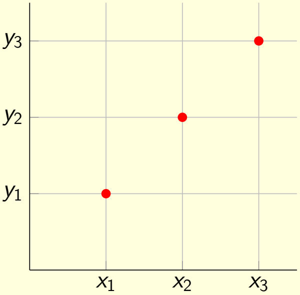

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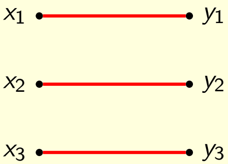

$$\widetilde{H}^0(G_\mu; \mathbb{C}) = \mathbb{C}\langle (1_{\{x_1\}}, 1_{\{y_1\}}), (1_{\{x_2\}}, 1_{\{y_2\}}), (1_{\{x_3\}}, 1_{\{y_3\}}) \rangle \Big/ \underbrace{\mathbb{C}\langle (1_X, 1_Y) \rangle}_{\text{pairs of constant random variables}}$$

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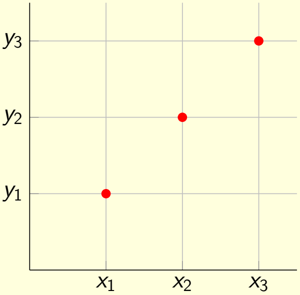

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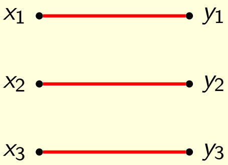

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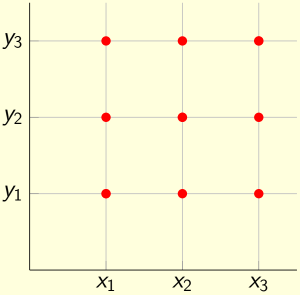

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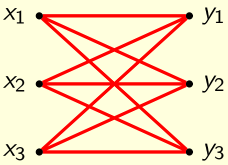

$\widetilde{H}^0(G_\mu; \mathbb{C}) \cong 0$

$$\mathrm{Supp}(\widehat{\mu}) = \mathrm{Supp}(\widehat{\mu}_X) \times \mathrm{Supp}(\widehat{\mu}_Y)$$

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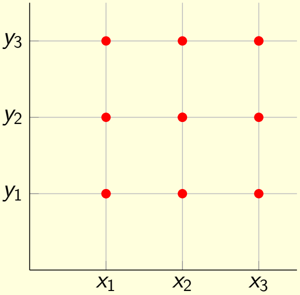

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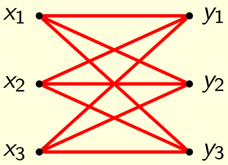

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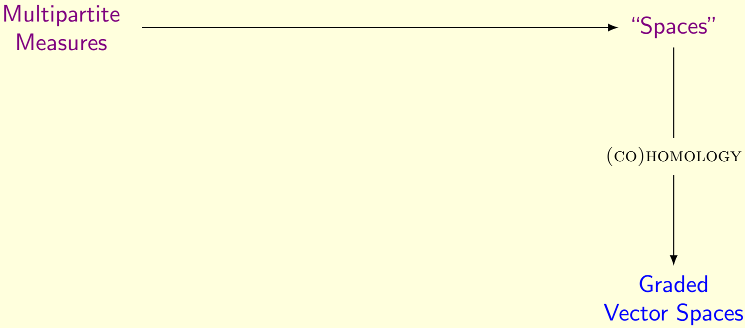
There are no *maximally* correlated pairs of random variables. But there are *statistically* correlated pairs if $\widehat{\mu} \neq \widehat{\mu}_X \times \widehat{\mu}_Y$.

What's the Big Idea? Numerical Invariants



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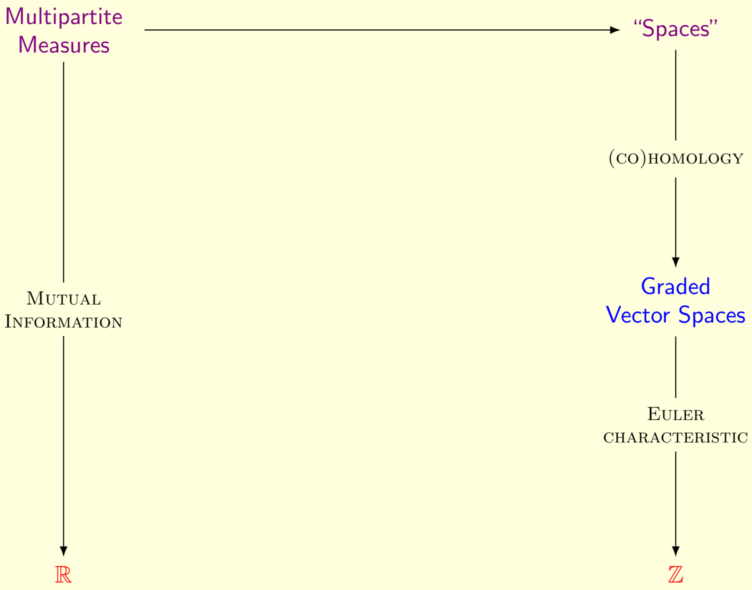


What's the Big Idea? Numerical Invariants

Multipartite Measures → "Spaces"

$$I_{AB} = S_A + S_B - S_{A\cup B}.$$

For N partite measures:

$$I_N = \sum_{k=0}^{N} (-1)^k \left[\sum_{|T|=k} S_T \right].$$

where the T are subsystems.

Mutual Information

(co)homology

Graded Vector Spaces

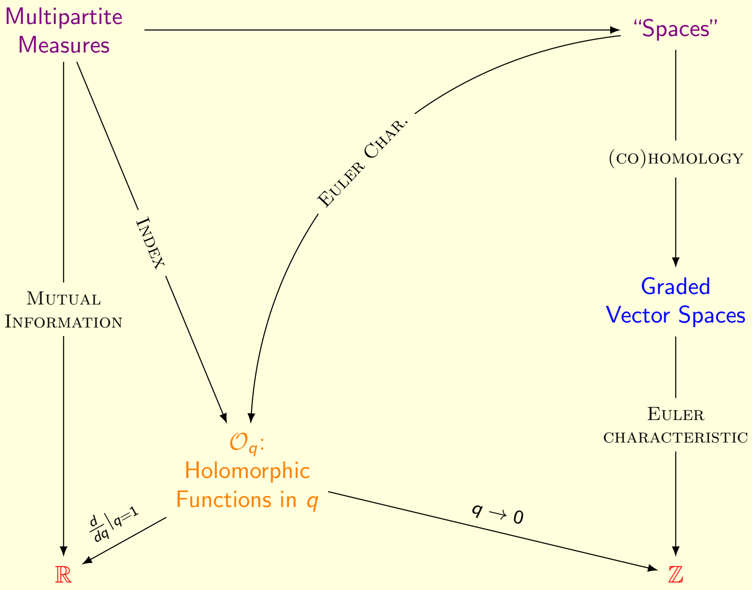
Euler characteristic

$$\chi\left(\bigoplus_{k=0}^{N} H^k \right) = \sum_{k=0}^{N} (-1)^k \dim(H^k)$$

$\mathbb{R}$

$\mathbb{Z}$

What's the Big Idea? Numerical Invariants



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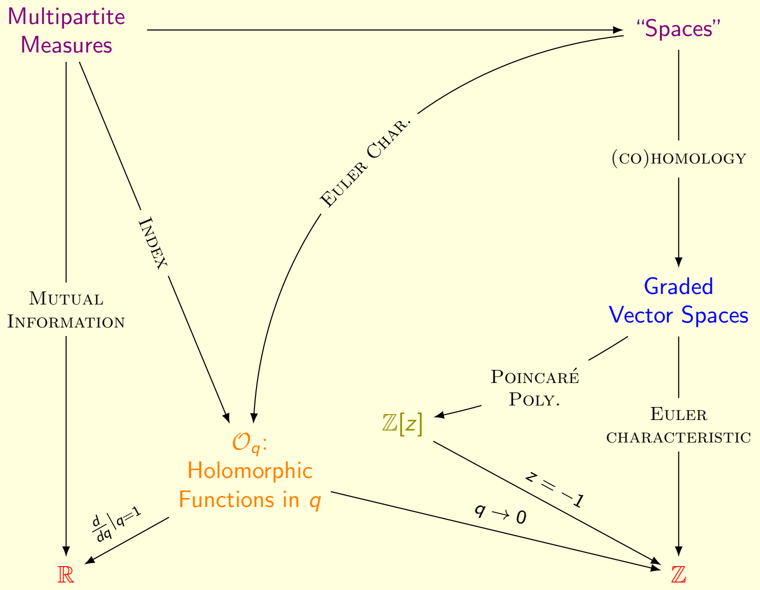
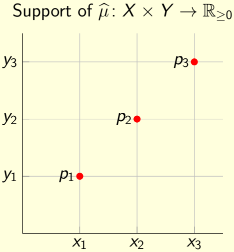
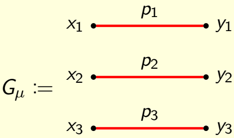


The Index of A Commutative Bipartite Measure





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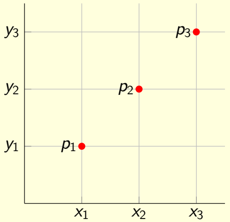
Support of $\widehat{\mu}\colon X \times Y \to \mathbb{R}_{\geq 0}$



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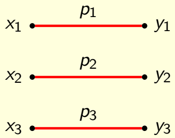

$$\mathfrak{X}_q[G_\mu] = \underbrace{\left(\sum_i p_i\right)^q}_{\dim_q \mu_\emptyset} - 2\underbrace{\sum_i (p_i)^q}_{\dim_q(\mu_X \boxplus \mu_Y)} + \underbrace{\sum_i (p_i)^q}_{\dim_q(\mu_{XY})} = -\substack{\text{Euler}\\ \text{Char}}[\mu_\emptyset \leftarrow \mu_X \boxplus \mu_Y \leftleftarrows \mu_{XY}]$$

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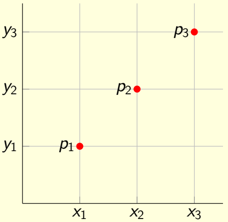

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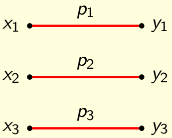

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$$\mathfrak{X}_0(G_\mu) = 1 - 2(3) + 3 = -2 = 1 - \chi(G_\mu) = -(\overbrace{\dim \widetilde{H}^0}^{2} - \overbrace{\dim \widetilde{H}^1}^{0})$$

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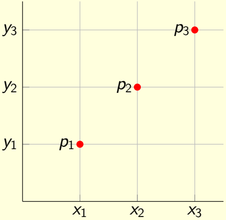

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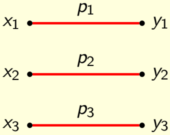

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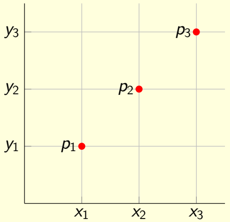
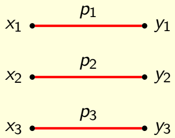
$$\left.\frac{d}{dq}\right|_{q=1} \mathfrak{X}_q(G_\mu) = \mathrm{Mass}(\mu)\log[\mathrm{Mass}(\mu)] - \sum_i p_i \log(p_i)$$

The Index of A Commutative Bipartite Measure

Support of $\widehat{\mu}\colon X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu :=$

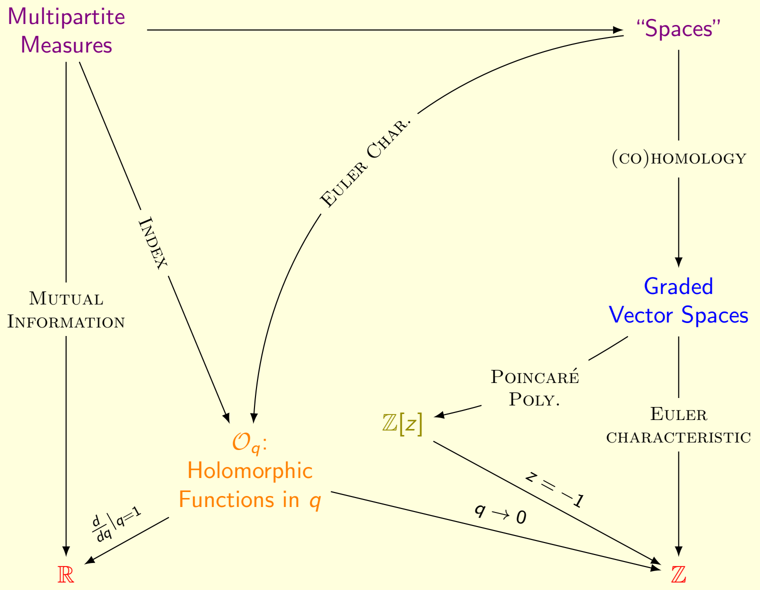


$$\mathfrak{X}_q[G_\mu] = \underbrace{\left(\sum_i p_i\right)^q}_{\dim_q \mu_\emptyset} - \underbrace{2\sum_i (p_i)^q}_{\dim_q(\mu_X \boxplus \mu_Y)} + \underbrace{\sum_i (p_i)^q}_{\dim_q(\mu_{XY})} = -{}^{\text{Euler}}_{\text{Char}}[\mu_\emptyset \leftarrow \mu_X \boxplus \mu_Y \leftleftarrows \mu_{XY}]$$

$$\mathfrak{X}_0(G_\mu) = 1 - 2(3) + 3 = -2 = 1 - \chi(G_\mu) = -(\overbrace{\dim \widetilde{H}^0}^{2} - \overbrace{\dim \widetilde{H}^1}^{0})$$

$$\frac{d}{dq}\bigg|_{q=1} \mathfrak{X}_q(G_\mu) = \mathrm{Mass}(\mu)\log[\mathrm{Mass}(\mu)] - \textstyle\sum_i p_i \log(p_i) = S_X + S_Y - S_{XY}$$

What's the Big Idea? Mysteries!



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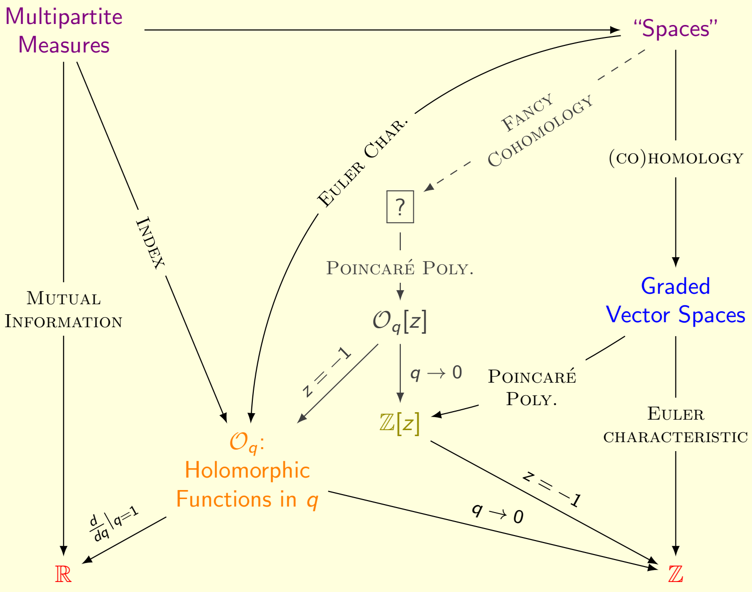


PART II
SELECT DETAILS
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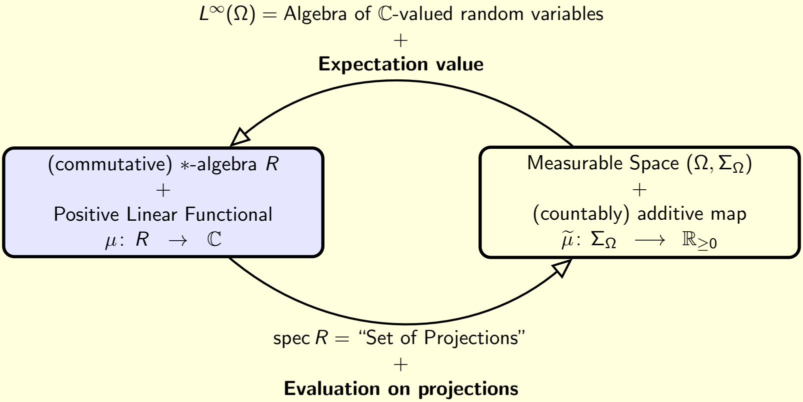
What's a Measure?



Positive means $\mu(r^* r) \geq 0$

See Dmitri Pavlov's *Gelfand-type duality for commutative von Neumann algebras*, arXiv:2005.05284.

What's a Measure?

$$\text{“measure”} = \text{(normal) positive linear functional on a } \overbrace{W^*\text{-algebra}}^{\text{“von Neumann algebra”}}\ R.$$

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Algebra R of Random Variables	Measure μ
$\mathrm{Fun}_{\mathbb{C}}(\Omega) \cong \mathbb{C}^{\lvert\Omega\rvert}$	$\mu(f) = \sum_{\omega\in\Omega} \widehat{\mu}_\omega f(\omega)$, $\widehat{\mu}_\omega \in \mathbb{R}_{\geq 0}$

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$\prod_{i=1}^n \mathrm{End}(\mathcal{H}_i)$	$\mu(r_1, \cdots, r_n) = \sum_i \mathrm{Tr}_{\mathcal{H}_i}[\widehat{\mu}^{(i)} r_i]$

The Category of Measures

Fix an algebra of random variables R with measures μ and ν:
$\mu \leq \nu$ if $\mu(r^* r) \leq \nu(r^* r)$ for all $r \in R$.

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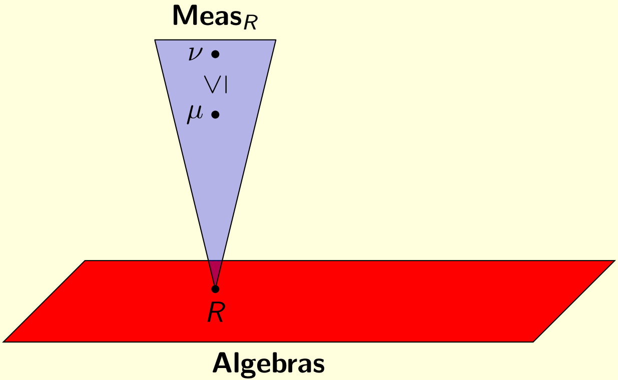
Definition

$\mathbf{Meas}_R$ is the category whose objects are measures $\mu\colon R \to \mathbb{C}$ and with a unique morphism $\mu \to \nu$ if $\mu \leq \nu$.

The Category of Measures



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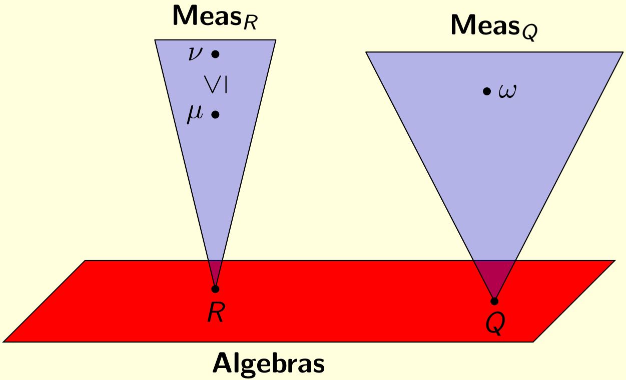
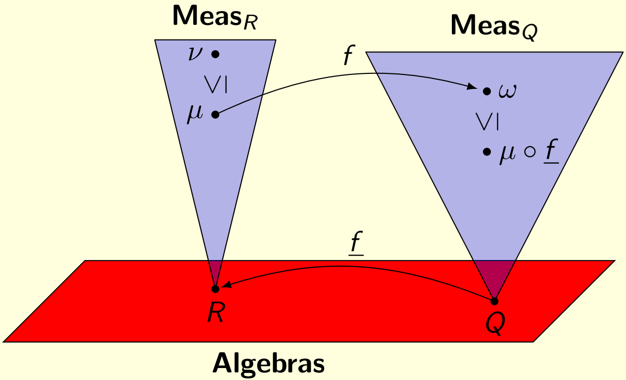
Meas is the category with objects given by measures (R, μ) for any R;



The Category of Measures

Definition

Meas is the category with objects given by measures (R, μ) for any R; a morphism $f\colon (R,\mu) \to (Q,\omega)$ given by an "underlying" homomorphism $\underline{f}\colon Q \to R$ such that $\mu \circ \underline{f} \leq \omega$.



The Category of Measures: Properties

Meas has:

- Coproducts ("disjoint union measures"):

$$\begin{aligned} \mu \boxplus \omega \colon R \times Q &\longrightarrow \mathbb{C} \\ (r, q) &\longmapsto \mu(r) + \omega(q) \end{aligned}$$

- Monoidal $\otimes$ products ("product measures"):

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- Rescaling: $(\mathbb{R}_{\geq 0}, \times) \to (\mathrm{End}(\mathbf{Meas}), \circ)$.
- The total mass functor:

$$\begin{aligned} \mathtt{Mass} \colon (\mathbf{Meas}, \otimes) &\longrightarrow (\mathbb{R}_{\geq 0}, \times) \\ \mu &\longmapsto \mu(1) \\ (\mu \leq \omega) &\longmapsto (\mu(1) \leq \omega(1)) \end{aligned}$$

The Category of Measures: Dimension

There is a homomorphism:

$$\dim\colon K_0(\mathbf{Meas}^{\mathrm{Fin}}) \longrightarrow \underbrace{\mathcal{O}(\mathbb{C})}_{\text{Holomorphic Functions on } \mathbb{C}}$$

$\dim(\mu \boxplus \nu) = \dim(\mu) + \dim(\nu)$ and $\dim(\mu \otimes \nu) = \dim(\mu)\dim(\nu)$.

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Note:

$$\dim_0(\mu) \coloneqq \lim_{q \to 0} \dim_q(\mu) = |\mathrm{Supp}(\widehat{\mu})| \in \mathbb{Z}_{\geq 0}.$$

The Category of Measures: the L^2 functor

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$L^2[\mathrm{Supp}(\widehat{\mu})]$ is the classical version of the Gelfand-Neumark-Segal representation of $L^\infty(\Omega)$ associated to μ. This is secretly a functor

$$\mathtt{GNS} : \mathbf{Meas}^{\mathrm{op}} \longrightarrow \mathbf{Rep}$$

Used in the construction of the cohomology of a measure.

Measure Families

A measure family over a measurable space (P, Σ_P) is a functor ("pre-cosheaf")

$$\mathtt{M}\colon\ \mathbf{\Sigma}_P \longrightarrow \mathbf{Meas}$$

where $\mathbf{\Sigma}_P$ is a category with:

- Objects given by measurable sets;
- A unique morphism $T \to V$ if $T \subseteq V$.

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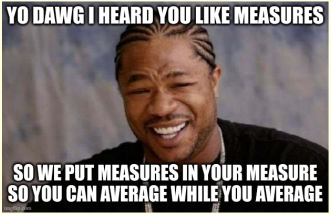
For every measurable subset of $T \subseteq P$ we have a measure $\mathtt{M}(T)\colon R_T \to \mathbb{C}$.



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Measure Families from Measures

Let $\widetilde{\mu}\colon \mathrm{Subsets}(\Omega) \to \mathbb{R}_{\geq 0}$ be a measure on a finite set Ω. There is a measure family:

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$$\begin{array}{ccc} \mathbf{Subsets}(\Omega) & \xrightarrow{\mathtt{A}^{\mu}} & \mathbf{Meas} \\ & \searrow^{\widetilde{\mu}} & \downarrow \mathtt{Mass} \\ & & \mathbb{R}_{\geq 0} \end{array}$$

By additivity: $\mathtt{A}^{\mu} \cong \boxplus_{\omega \in \Omega} \mathtt{A}^{\mu}|_{\{\omega\}}$:

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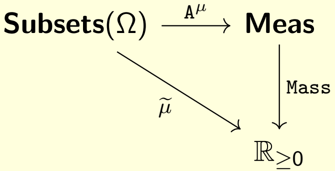

By additivity: $\mathtt{A}^{\mu} \cong \boxplus_{\omega\in\Omega} \mathtt{A}^{\mu}|_{\{\omega\}}$: cosheaf-like: global data comes from "additively" gluing together local data.

Measure Families from Multipartite Measures

$$\text{bipartite measure } \boldsymbol{\mu} = \begin{matrix} R_{\mathsf{A}},\ R_{\mathsf{B}} \text{ a pair of algebras} \\ + \\ \mu : R_{\mathsf{A}} \otimes R_{\mathsf{B}} \longrightarrow \mathbb{C} \text{ a measure} \end{matrix}$$

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Giving us the reduced measures ("partial traces" / "marginal measures")

$$\begin{aligned} \mu_{\mathsf{A}} \coloneqq \mu \circ \epsilon_{\mathsf{A}} : R_{\mathsf{A}} &\longrightarrow \mathbb{C} \\ a &\longmapsto \mu(a \otimes 1) \end{aligned} \qquad \begin{aligned} \mu_{\mathsf{B}} \coloneqq \mu \circ \epsilon_{\mathsf{B}} : R_{\mathsf{B}} &\longrightarrow \mathbb{C} \\ b &\longmapsto \mu(1 \otimes b) \end{aligned}$$

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But this isn't a measure family: it isn't covariant out of **Subset**(P)! To get a measure family, we define:

$$\mathtt{M}^{\boldsymbol{\mu}} \coloneqq \mathtt{C}_{\boldsymbol{\mu}} \circ \underbrace{(-)^c}_{\substack{\text{complementation} \\ \text{functor}}}\colon \textbf{Subsets}(P) \longrightarrow \textbf{Meas}$$

Toward (Semi-)Simplicial Measures

Let $\mathtt{M}$ be a measure family over the set $\{\mathsf{A}, \mathsf{B}\}$. We have a diagram in **Meas**:

$$\mathtt{M}(\{\mathsf{A},\mathsf{B}\}) \longleftarrow \mathtt{M}(\{\mathsf{A}\}) \boxplus \mathtt{M}(\{\mathsf{B}\}) \leftleftarrows \mathtt{M}(\emptyset)$$

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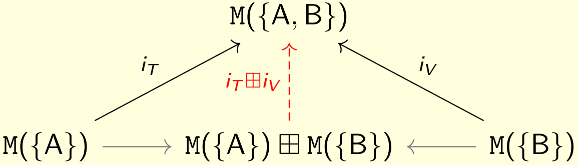
These arrows use the coproduct property of $\boxplus$, e.g.



Čech (Semi)-Simplicial Measures from a Cover

For $\mathtt{M}$ a measure family over P: let $\{U_i\}_{i \in I}$ be a measurable cover of P. Define $U_J := \bigcap_{j \in I} U_j$ and $U_\emptyset = P$. We have (Note: $U_J \supseteq U_K$ if $J \subseteq K$):

$$\mathtt{M}(U_\emptyset) \longleftarrow \bigoplus_{|J|=1} \mathtt{M}(U_J) \leftleftarrows \cdots \begin{matrix} \longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow \end{matrix} \bigoplus_{|J|=|I|-1} \mathtt{M}(U_J) \begin{matrix} \longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow \end{matrix} \mathtt{M}(U_I)$$

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Special Case: the complementary cover $\{p^c\}_{p\in P}$ of P:

$$\mathtt{M}(\emptyset^c) \longleftarrow \bigoplus_{|T|=1} \mathtt{M}(T^c) \overset{\longleftarrow}{\longleftarrow} \cdots \begin{matrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{matrix} \bigoplus_{|T|=|P|-1} \mathtt{M}(T^c) \begin{matrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{matrix} \mathtt{M}(P^c)$$

Čech (Semi)-Simplicial Measures from a Cover

$$\mathtt{M}(\emptyset^c) \longleftarrow \bigoplus_{|T|=1} \mathtt{M}(T^c) \leftleftarrows \cdots \begin{matrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{matrix} \bigoplus_{|T|=|P|-1} \mathtt{M}(T^c) \begin{matrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{matrix} \mathtt{M}(P^c)$$

When $\mathtt{M} = \mathtt{M}^{\boldsymbol{\mu}}$ for a multipartite measure $\boldsymbol{\mu}$, the above becomes:

$$\overbrace{\mu_\emptyset}^{\text{Deg. } -1} \longleftarrow \overbrace{\bigoplus_{|T|=1} \mu_T}^{\text{Deg. } 0} \leftleftarrows \cdots \begin{matrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{matrix} \overbrace{\bigoplus_{|T|=|P|-1} \mu_T}^{\text{Deg. } |P|-2} \underbrace{\begin{matrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{matrix}}_{\text{“Partial Traces”}} \overbrace{\mu_P}^{\text{Deg. } |P|-1}$$

There's now a few things we can extract from our (semi-)simplicial measure:

- Simplicial complexes
- Cohomology/chain complexes of vector spaces
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$$\mathfrak{X}(\mathtt{M}) := -\underset{\text{Char}}{\text{Euler}}\left[\mathtt{M}(\emptyset^c) \longleftarrow \bigoplus_{|T|=1} \mathtt{M}(T^c) \leftleftarrows \cdots \begin{smallmatrix}\longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow\end{smallmatrix} \bigoplus_{|T|=|P|-1} \mathtt{M}(T^c) \begin{smallmatrix}\longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow\end{smallmatrix} \mathtt{M}(P^c)\right]$$

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Some Properties of the Index

$$\mathfrak{X}(\mathtt{M}) = \sum_{T \subseteq P} (-1)^{|T|} \dim \left[\mathtt{M}(T^c) \right].$$

- When $q = 0$, $\mathfrak{X}_0(\mathtt{M}) \in \mathbb{Z}$. If $\mathtt{M} = \mathtt{M}^{\boldsymbol{\mu}}$ for $\boldsymbol{\mu}$ commutative:

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- $\frac{d}{dq}\Big|_{q=1} \mathfrak{X}(\mathtt{M}^{\boldsymbol{\mu}})$ is multipartite mutual information:

$$I(\boldsymbol{\mu}) = \sum_{T \subseteq P} (-1)^{|T|-1} S(\mu_T) = \frac{d}{dq}\Big|_{q=1} \mathfrak{X}(\mathtt{M}^{\boldsymbol{\mu}}).$$

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- $\frac{1}{q-1} \mathfrak{X}_q(\mathtt{M}^{\boldsymbol{\mu}})$ is Tsallis-deformed multipartite mutual information.

Some Properties of the Index

$$\mathfrak{X}(\mu) = \sum_{T \subseteq P} (-1)^{|T|} \dim \left[\mathtt{M}(T^c) \right]$$

Theorem

$\mathfrak{X}$ defines a homomorphism $K_0(\mathbf{MeasFam}^{\mathrm{fin}}) \to \mathcal{O}(\mathbb{C})$: i.e.

- $\mathfrak{X}(\mathtt{M} \otimes \mathtt{N}) = \mathfrak{X}(\mathtt{M})\mathfrak{X}(\mathtt{N})$
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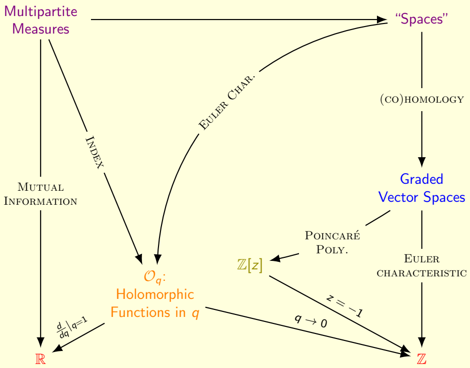
Theorem

$\mathfrak{X}(\mathtt{M}) = 0$ if $|P| \geq 2$ and $\mathtt{M} \cong \mathtt{M}|_T \boxplus \mathtt{M}|_V$ for any $T, V \subseteq P$.

So $\mathfrak{X}(\mathtt{M})$ detects the failure of *additive* ($\boxplus$) descent of data, while $\frac{d}{dq}\Big|_{q=1} \mathfrak{X}(\mathtt{M})$ detects the failure of *multiplicative* ($\otimes$) descent.

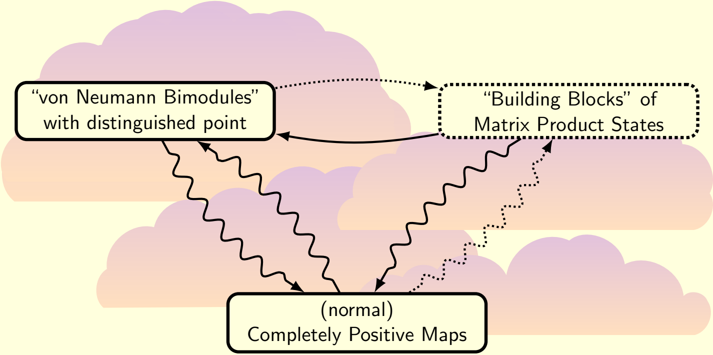
Summary

- Mutual information (and its deformations) emerge naturally as an Euler characteristic (the "index") of some emergent "space".
- Random variables capturing "maximal" non-local correlations between subsystems are captured by cohomology.



Spinoff Work

With Roman Geiko and Greg Moore:



Wiggly arrow: Equivalence with reflective subcategory on top row. Dotted: Requires the data of Morita equivalent W^*-algebras.