The Secret Topological Life of Shared Information

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July 28, 2020

What's the Big Idea?

Multipartite

State

- $\psi \in \bigotimes_{s \in P} \mathcal{H}_s$
- $\hat{\rho} \in \mathrm{Dens}\left(\bigotimes_{s \in P} \mathcal{H}_s\right)$
- $\mu : \prod_{s \in P} \Omega_s \to \mathbb{R}_{\geq 0}$

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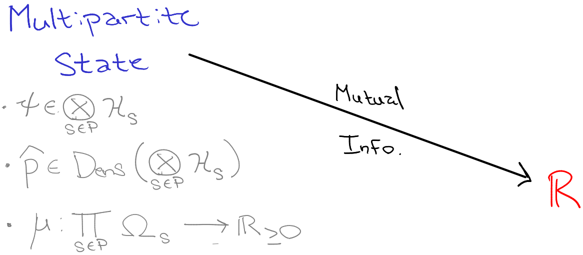

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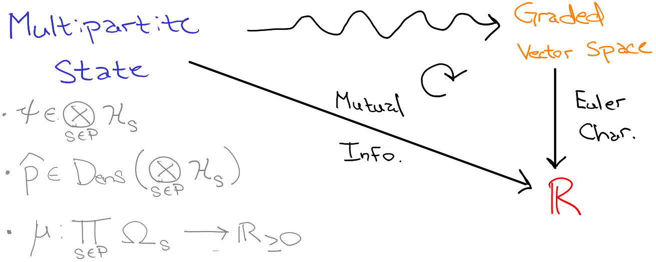

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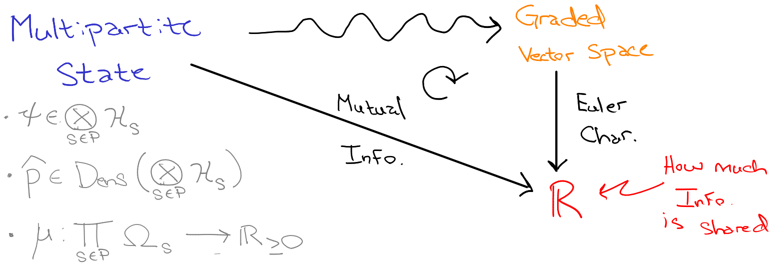

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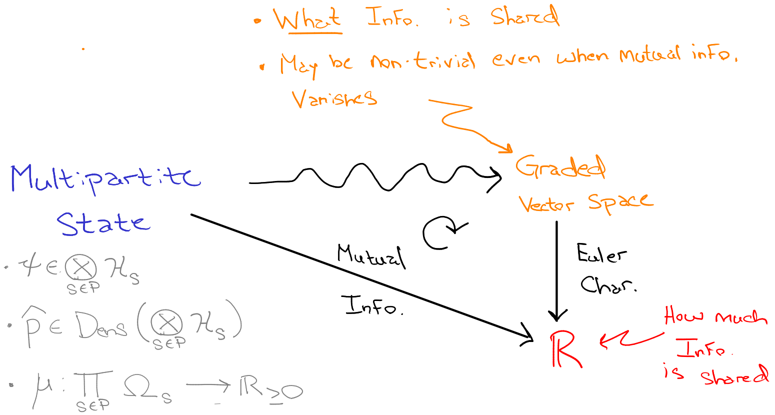

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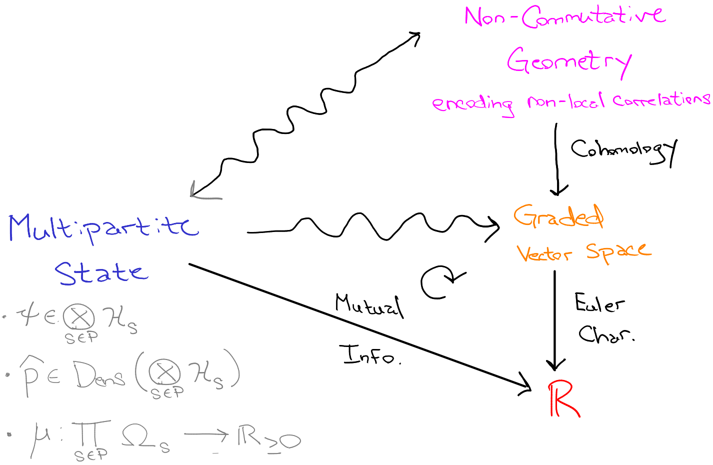

Why is this cool?

$$N\text{-partite state} \rightsquigarrow \bigoplus_{k=0}^{N-1} H^k\left[N\text{-partite state}\right]$$

$$H^k\left[N\text{-partite state}\right] = \left\{\begin{array}{c}\text{tuples of } (k+1)\text{-body operators}\\ \text{exhibiting correlations}\end{array}\right\} \Big/ \left\{\begin{array}{c}\text{trivial}\\ \text{correlations}\end{array}\right\},\ k < N-1$$

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$$\left[\left(\left|0_\mathrm{A}\right\rangle\left\langle 0_\mathrm{A}\right|, \left|0_\mathrm{B}\right\rangle\left\langle 0_\mathrm{B}\right|\right)\right] \in H^0\left(\left|0_\mathrm{A}0_\mathrm{B}\right\rangle + \left|1_\mathrm{A}1_\mathrm{B}\right\rangle\right)$$

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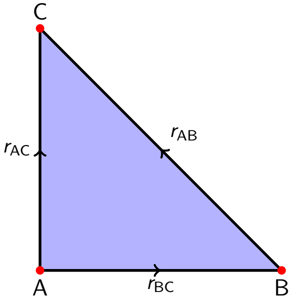
"r_A and r_B are maxmly. correlated"

1-cochains for a tripartite state

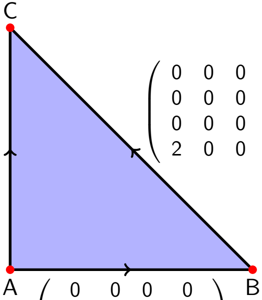
$$\psi \in \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}} \otimes \mathcal{H}_{\mathsf{C}}$$



$$[(r_{\mathsf{AB}}, r_{\mathsf{AC}}, r_{\mathsf{BC}})] \in H^1(\psi) \iff \widetilde{r}_{\mathsf{BC}} + \widetilde{r}_{\mathsf{AB}} \underset{\mathsf{ABC}}{\sim} \widetilde{r}_{\mathsf{AC}}$$

1-cochains for the GHZ state

$$\psi = |\mathsf{GHZ}\rangle_3 = |0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$$



$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$I_n = \sum_{T \subseteq P} (-1)^{|T|} S_T \in \mathbb{R}$$

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while for $\bullet = 0,\ 1$

$$H^\bullet[|000\rangle + |111\rangle] \neq 0,$$
$$H^\bullet[|001\rangle + |010\rangle + |100\rangle] \neq 0$$

Slide showing Non-Commutative Geometry $\rightsquigarrow$ State Index as an Euler characteristic.

$$\rho_P \in \mathrm{Dens}(\otimes_{p \in P} \mathcal{H}_p)$$

State Index $\in \mathcal{O}(\mathbb{C}_\alpha \times \mathbb{C}_q \times \mathbb{C}_r)$

$\alpha \to 0$

$\times \frac{1}{r(q-1)}$

$q \to 0$

$\alpha,\ r \in \mathbb{Z}$

Tsallis/Rényi Deformed Mutual Information $\in \mathcal{O}(\mathbb{C}_q \times \mathbb{C}_r)$

Euler Characteristics of Complexes of Vector Spaces $\in \mathbb{Z}$

$q \to 1$

Mutual Information $\in \mathbb{R}$

Before We Define Things

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Possibly new link invariants: $L \subset S^3$ a link with N-components;[1]

$$\psi_L := \mathcal{Z}_{\mathrm{CS}}[S^3 - L] \in \mathcal{Z}_{\mathrm{CS}}[\mathbb{T}]^{\otimes N}$$

Corresponding cohomology, Poincaré polynomials, and state indices are frame-equivariant/independent *link* invariants.

[1] Based on conversations with G. Moore. See work of Swingle and Balasubramanian, et. al.

Similar/Related work

What's a state?

$$\text{“state”} = (\text{normal}) \text{ positive linear functional on a } \overbrace{W^*\text{-algebra}}^{\text{“von Neumann algebra”}} R.$$

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$\prod_{i=1}^n \mathrm{End}(\mathcal{H}_i)$	$\rho(r_1, \cdots, r_n) = \sum_i \mathrm{Tr}_{\mathcal{H}_i}[\widehat{\rho}^{(i)} r_i]$

$$\underset{\prod_i \mathrm{End}(\mathcal{H}_i)}{\text{State on}} \leftrightarrow \underset{(\widehat{\rho}^{(1)}, \cdots, \widehat{\rho}^{(n)})}{\text{Tuple of density states}}$$

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State on $\prod_i \mathrm{End}(\mathcal{H}_i)$ $\leftrightarrow$ Tuple of density states $(\widehat{\rho}^{(1)}, \cdots, \widehat{\rho}^{(n)})$

State on $\prod_{i=1}^n \mathbb{C}$ $\leftrightarrow$ Tuple of non-negative reals $(\mu^{(1)}, \cdots, \mu^{(n)})$

What's a Bipartite State? (roughly)

$$\text{"bipartite state"} = \begin{array}{c} R_\mathsf{A},\ R_\mathsf{B} \text{ a pair of algebras} \\ + \\ \rho : R_\mathsf{A} \otimes R_\mathsf{B} \longrightarrow \mathbb{C} \text{ a state} \end{array}$$

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We have homomorphisms

$$\begin{aligned} \epsilon_{\mathsf{A}} : R_{\mathsf{A}} &\longrightarrow R_{\mathsf{A}} \otimes R_{\mathsf{B}} \\ a &\longmapsto a \otimes 1 \end{aligned} \qquad \begin{aligned} \epsilon_{\mathsf{B}} : R_{\mathsf{B}} &\longrightarrow R_{\mathsf{A}} \otimes R_{\mathsf{B}} \\ b &\longmapsto 1 \otimes b \end{aligned}$$

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Giving us the reduced states (“partial traces” / “partial measures”)

$$\begin{aligned} \rho_{\mathsf{A}} := \rho \circ \epsilon_{\mathsf{A}} : R_{\mathsf{A}} &\longrightarrow \mathbb{C} \\ a &\longmapsto \rho(a \otimes 1) \end{aligned} \qquad\qquad \begin{aligned} \rho_{\mathsf{B}} := \rho \circ \epsilon_{\mathsf{B}} : R_{\mathsf{B}} &\longrightarrow \mathbb{C} \\ b &\longmapsto \rho(1 \otimes b) \end{aligned}$$

What is Factorizability?

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What's a Multipartite State?

$$\text{“multipartite state”} \quad \text{“}=\text{”} \quad \begin{array}{c} (R_p)_{p\in P} \text{ tuple of algebras} \\ + \\ \rho : \bigotimes_{p\in P} R_p \longrightarrow \mathbb{C} \text{ a state} \end{array}$$

For any subset $T \subseteq P$ we have algebras $R_T := \bigotimes_{t\in T} R_t$ $(R_\emptyset = \mathbb{C})$, and maps

$$\epsilon_T : R_T \longrightarrow R_P$$

Define the reduced states

$$\rho_T := \rho \circ \epsilon_T : R_T \to \mathbb{C}$$

Why Geometry? Homological Obstructions, that's why

$$\rho : R_{\mathsf{A}} \otimes R_{\mathsf{B}} \longrightarrow \mathbb{C}$$

Factorizability

Descent of data to subsystems: all global data comes from gluing local data: $\rho(\sum_{ij} r^i_{\mathsf{A}} \otimes r^j_{\mathsf{B}}) = \frac{1}{\rho(1)} \sum_{ij} \rho_{\mathsf{A}}(r^i_{\mathsf{A}})\rho_{\mathsf{B}}(r^j_{\mathsf{B}})$.

Failure to Factorize

Obstruction to descent: $\rho(r_{\mathsf{A}} \otimes r_{\mathsf{B}}) \neq \frac{1}{\rho(1)}\rho_{\mathsf{A}}(r_{\mathsf{A}})\rho_{\mathsf{B}}(r_{\mathsf{B}})$ for some $(r_{\mathsf{A}}, r_{\mathsf{B}})$

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Homological Alarm Bells!

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"$H^0(\rho) = \{(r_{\mathsf{A}}, r_{\mathsf{B}}) \in R_{\mathsf{A}} \times r_{\mathsf{B}} : \rho(1)\rho(r_{\mathsf{A}} \otimes r_{\mathsf{B}}) \neq \rho_{\mathsf{A}}(r_{\mathsf{A}})\rho_{\mathsf{B}}(r_{\mathsf{B}})\}$"

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Factorizability ⟷ Descent of data to subsystems: all global data comes from gluing local data: $\rho(\sum_{ij} r^i_{\mathsf{A}} \otimes r^j_{\mathsf{B}}) = \frac{1}{\rho(1)} \sum_{ij} \rho_{\mathsf{A}}(r^i_{\mathsf{A}})\rho_{\mathsf{B}}(r^j_{\mathsf{B}})$.

⇕

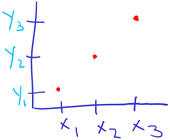
Failure to Factorize ⟷ Obstruction to descent: $\rho(r_{\mathsf{A}} \otimes r_{\mathsf{B}}) \neq \frac{1}{\rho(1)} \rho_{\mathsf{A}}(r_{\mathsf{A}})\rho_{\mathsf{B}}(r_{\mathsf{B}})$ for some $(r_{\mathsf{A}}, r_{\mathsf{B}})$

$$H^0[\rho] = \{(r_{\mathsf{A}}, r_{\mathsf{B}}) \in R_{\mathsf{A}} \times R_{\mathsf{B}} : r_{\mathsf{A}} \text{ and } r_{\mathsf{B}} \text{ are mxmly. correlated}\}/\mathbb{C}\langle(1,1)\rangle$$

Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$$\mu : X \times Y \to \mathbb{R}_{\geq 0}$$



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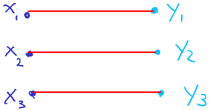

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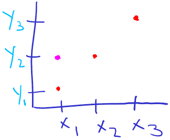

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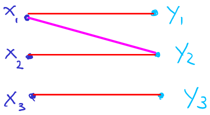

Multipartite Measures

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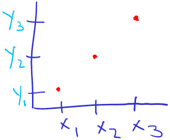
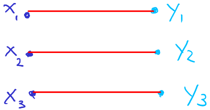


Commutative Geometry

encoding non-local correlations

$G_\mu =$



$H^0(G_\mu; \mathbb{C}) \cong \mathbb{C}^3$

Multipartite Measures $\longleftrightarrow$ Commutative Geometry encoding non-local correlations

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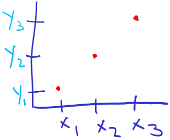

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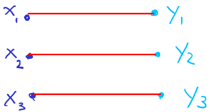

$$H^0(G_\mu ; \mathbb{C}) = \mathbb{C}\langle (1_{x_1}, 1_{y_1}), (1_{x_2}, 1_{y_2}), (1_{x_3}, 1_{y_3}) \rangle$$

1_z = indicator function on pt. z

= Pairs of "non-locally", maximally correlated random variables

Multipartite Measures $\longleftrightarrow$ Commutative Geometry

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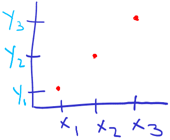

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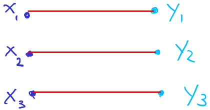

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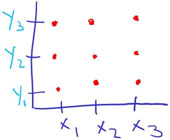
$$\tilde{H}^0(G_\mu; \mathbb{C}) = H^0(G_\mu; \mathbb{C}) \,/\, \mathbb{C}\langle (\textstyle\sum_i 1_{x_i}, \sum_i 1_{y_i}) \rangle$$

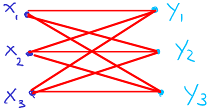
"non-trivial" "non-local" maximal correlations

Pairs of constant random vars.

Multipartite Measures $\longleftrightarrow$ Commutative Geometry encoding non-local correlations

$$\mu : X \times Y \to \mathbb{R}_{\geq 0}$$



$G_\mu =$ 

$$H^0(G_\mu; \mathbb{C}) \cong \mathbb{C}$$

$$\tilde{H}^0(G_\mu; \mathbb{C}) = 0$$

Mutual Info. as an Euler Char.

Mutual Information:

$$I_2(\rho_{\mathsf{AB}}) = S(\rho_{\mathsf{A}}) + S(\rho_{\mathsf{B}}) - S(\rho_{\mathsf{AB}})$$

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Recall: States on $\prod_{i=1}^{n} \mathrm{End}(\mathcal{H}_i) \longleftrightarrow$ tuples of density states $(\widehat{\rho}^{(1)}, \cdots, \widehat{\rho}^{(n)})$

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$$S[(\widehat{\rho}^{(1)}, \cdots, \widehat{\rho}^{(n)})] = \sum_{i=1}^{n} \underbrace{\mathrm{Tr}[\widehat{\rho}^{(i)} \log \widehat{\rho}^{(i)}]}$$

Multipartite Mutual information:

$$I_{|P|}(\rho_P) = \sum_{\emptyset T \subseteq P} (-1)^{|T|} S(\rho_T)$$

$$I_{|Q \cup R|}(\rho_Q \otimes \rho_R) = 0$$

Why Geometry? Mutual Info looks like an Euler Characteristic

C a sufficiently nice $\otimes$-category with all coproducts $\oplus$

An Euler characteristic (valued in a ring R) is an assignment that takes in any object A of **C** and outputs $\chi(A) \in R$ such that:

$\chi(A)$ only depends on A up to iso.

$\chi(A \oplus B) = \chi(A) + \chi(B)$

$\chi(A \otimes B) = \chi(A)\chi(B)$

equivalently χ is a homomorphism

$$\chi : K_0(\mathbf{C}) \to R$$

for some ring R.

$$I(\rho_P \otimes \varphi_Q) \equiv 0 \text{ not } I_P(\rho_P) I(\varphi_Q)$$

The GNS Functor

Recall the GNS construction:

$$\rho : A \to \mathbb{C} \xmapsto{\texttt{GNS}_A} (L^2_\rho[A/\mathfrak{I}_\rho], [1])$$

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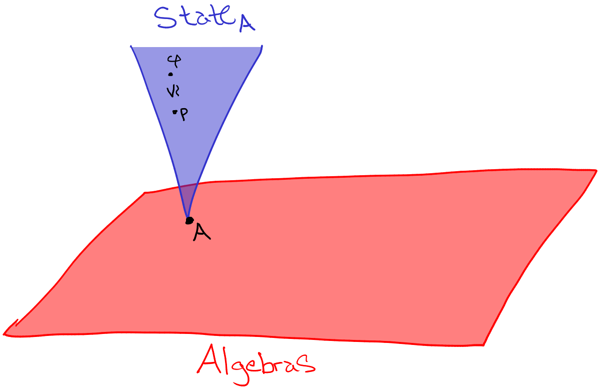
This is secretly a functor

$$\texttt{GNS}_A : \textbf{State}_A \to \textbf{Rep}_A$$

	$\textbf{State}_A$	$\textbf{Rep}_A$
Objects	Positive linear funls $\rho : R \longrightarrow \mathbb{C}$	$*$-representations of A
Morphisms	$\rho \longrightarrow \varphi$ $\Updownarrow$ $\rho \leq C\varphi$ for some $C > 0$	(bounded) intertwiners

The Category of States



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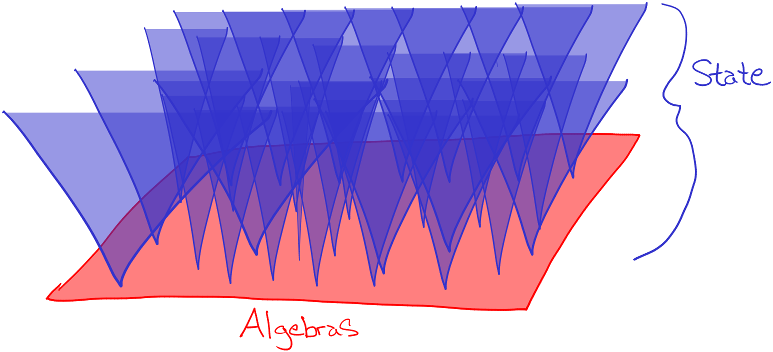

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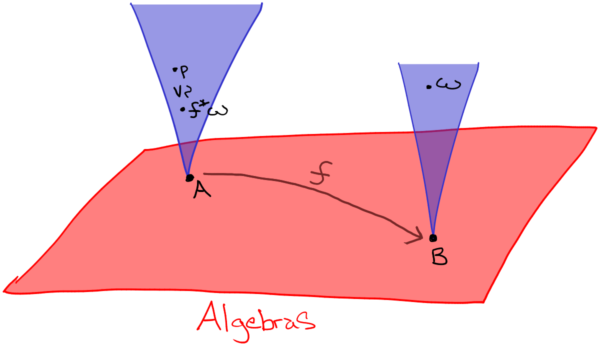

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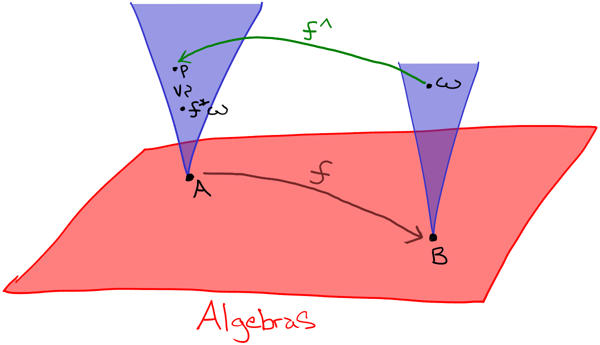

The Category of States

$$\texttt{GNS} : \textbf{State}^{\mathrm{op}} \longrightarrow \textbf{Rep}$$

	State	**Rep**
Objects	(R, ρ)	Algebras and "left modules" $(R, {}_R M)$
Morphisms	"duals" of algebra maps playing nicely with states	Algebra maps + intertwiners playing nicely together
(co)products	Classical sum $(A, \rho) \boxplus (B, \varphi) = (A \times B, \rho \times \varphi)$	Products $(A, M) \times (B, N) =$ $(A \times B, M \times N)$

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$$\texttt{GNS}(\boxplus) = \prod$$

(Non-Comm.) Geometry from a Multipartite State

A multipartite state over a finite set P is a functor

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Can make this covariant using complementation on sets, then use Čech theory to construct a “simplicial state”

$$\underbrace{\rho_\emptyset}_{(\mathbb{C},\rho(1))} \longleftarrow \boxplus_{|T|=1} \rho_T \leftleftarrows \boxplus_{|T|=2} \rho_T \; \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \cdots \underbrace{\substack{\longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow}}_{N-1 \text{ arrows}} \boxplus_{|T|=N-1} \rho_T \underbrace{\substack{\longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow}}_{\substack{N \text{ arrows} \\ \text{“partial traces”}}} \rho_P$$

(Non-Comm.) Geometry from a Multipartite State

$$\rho_\emptyset \longleftarrow \underset{|T|=1}{\boxplus} \rho_T \leftleftarrows \underset{|T|=2}{\boxplus} \rho_T \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \cdots \;\substack{\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow}\; \underset{|T|=N-1}{\boxplus} \rho_T \;\substack{\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow}\; \rho_P$$

$$\Big\downarrow \; \mathtt{GNS}$$

$$\underbrace{\mathtt{GNS}(\rho_\emptyset)}_{\mathbb{C}\mathbb{C}} \longrightarrow \prod_{|T|=1} \mathtt{GNS}(\rho_T) \rightrightarrows \prod_{|T|=2} \mathtt{GNS}(\rho_T) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow}\; \prod_{|T|=N-1} \mathtt{GNS}(\rho_T) \;\substack{\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow}\; \mathtt{GNS}(\rho_P)$$

Forget Algebra
+Alternating sum
of arrows

$$0 \to \mathbb{C} \xrightarrow{d^{-1}} \prod_{|T|=1} \mathtt{GNS}(\rho_T) \xrightarrow{d^{0}} \prod_{|T|=2} \mathtt{GNS}(\rho_T) \xrightarrow{d^{1}} \cdots \xrightarrow{d^{N-2}} \prod_{|T|=N-1} \mathtt{GNS}(\rho_T) \xrightarrow{d^{N-1}} \mathtt{GNS}(\rho_P) \to 0$$

Summary

Mutual information (and its deformations) of a multipartite state emerge naturally from the Euler characteristic (the "state index") of some canonically associated non-commutative "space."

The precise operators/random variables capturing non-local correlations are captured by cohomology.

Similar to how cohomology is a finer invariant than an Euler characteristic, cohomology is finer than mutual information.

Software

Software computing cohomology/Poincaré polynomials is available at `github.com/tmainero`.