RESEARCH STATEMENT

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1. Overview

My interests primarily reside at the intersection of quantum field theory, geometry, and quantum information (in the broad senses of those words). While my Ph.D. is in physics and I have experience in high-energy theory, I am also motivated by questions of geometry/topology, using physics as an inspiration. My most recent work [1] involves the application of techniques from topology/geometry—most notably homological techniques—to study quantum (and classical) information theoretic quantities.

2. Entanglement, Cohomology, and the Categorification of Mutual Information

Consider the following factorization questions:

(C) Suppose we are handed a measure μ on a product of measurable spaces (sets each equipped with a σ-algebra of "measurable subsets") $\mathbb{X}_1 \times \mathbb{X}_2 \cdots \times \mathbb{X}_n$. Does $\mu = \mu_1 \times \cdots \times \mu_n$ for some μ_i a measure on $\mathbb{X}_i$?

(Q) Suppose we are handed anon-zero element ψ in a tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. Does $\psi = \psi_1 \otimes \cdots \otimes \psi_n$ for some $\psi_i \in \mathcal{H}_i$?

The first question is one of classical probability theory: one is asking if random variables associated to $\mathbb{X}_i$—e.g. measurable functions $f : \mathbb{X}_i \to (\mathbb{C}, \mathrm{Borel}(\mathbb{C}))$—are independent of random variables associated to $\mathbb{X}_j$ for $j \neq i$. The second question is quantum mechanical in nature: states that fail to factorize in such a way are called *entangled*, and play a fundamental role in quantum information/computation [2, 3]. Entangled states are also a major focus in high-energy physics, particularly after the work of Ryu and Takayanagi [4], which relates entanglement entropy (a numerical measure of entanglement) of states in a conformal field theory to the area of minimal surfaces in a gravity dual theory. Some of these ideas have discrete, computationally accessible versions [5, 6, 7] using the techniques of tensor networks [8, 9, 10] and error correcting codes.

Questions (C) and (Q) are special cases of the following:

(QC) Suppose we are handed a *multipartite expectation value*: a (positive) linear functional $\mathbb{E} : \mathcal{A} \to \mathbb{C}$ on an algebra of random variables $\mathcal{A}$ (a $*$-algebra) that factorizes as a tensor product $\mathcal{A} = \bigotimes_{i=1}^{n} \mathcal{A}_i$. Does $\mathbb{E}$ factorize as a tensor product of expectation values on each $\mathcal{A}_i$?

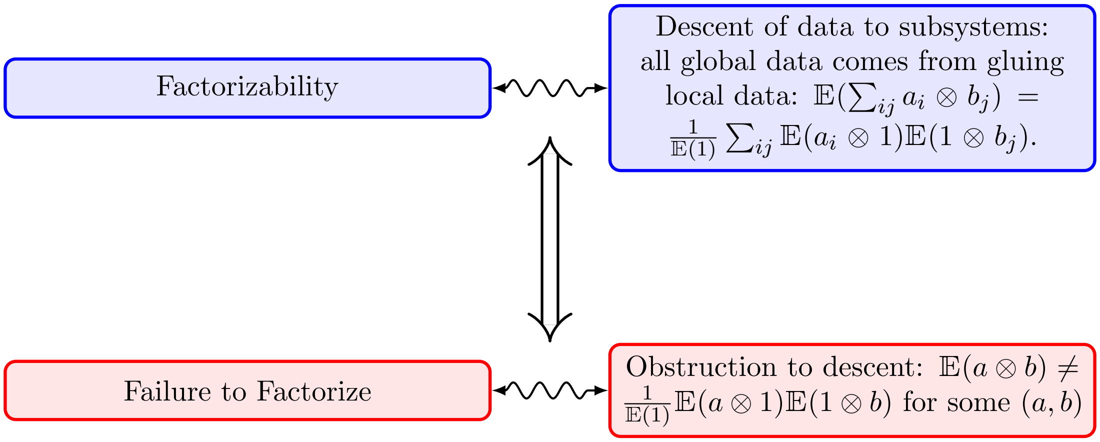
We can recover question (C) by taking $\mathcal{A}$ to be the (commutative) algebra of essentially bounded $\mathbb{C}$-valued functions on a product of measure spaces; to recover question (Q) we take $\mathcal{A}$ to be the algebra of bounded endomorphisms—"operators"—on a tensor product of Hilbert spaces.[1] More generally it is natural to take $\mathcal{A}$ to be a W^*-algebra: a special type of C^*-algebra that abstracts von Neumann algebras.[2]

A recent focus of my research is on a cohomological way of studying question (QC): to every multipartite expectation value there are cochain complexes whose cohomologies detect the failure

[1] In the former case $\mathbb{E}$ is given by integration of essentially bounded functions with respect to the measure; in the latter $\mathbb{E}(a) = \langle \psi, a\psi \rangle$ where ψ is a state and $\langle -, - \rangle$ the hermitian inner product.

[2] The study of expectation values on a W^*-algebra with non-trivial center (e.g. the Cartesian product of matrix algebras) can be thought of as the study of mixed quantum-classical systems.

of the expectation value to factorize. The intuition behind the existence of such cohomologies is provided by the following diagram.



Cohomology is precisely formulated to detect obstructions of this form. I was able to realize the above idea in [1]: given a multipartite expectation value, one can construct cochain complexes of $\mathbb{C}$-vector spaces[3] using ideas related to the Gelfand-Neumark-Segal construction of $*$-representations. Given an N-partite expectation value $\mathbb{E} : \bigotimes_{p \in P} \mathcal{A}_i \longrightarrow \mathbb{C}$ (where P is a finite ordered set) there are chain complexes of the form

$$\cdots \to 0 \to \mathbb{C} \to \underbrace{\prod_{\{T \subseteq P : |T|=1\}} S_T(\mathbb{E})}_{\mathrm{C}^0(\mathbb{E})} \to \cdots \to \underbrace{\prod_{\{T \subseteq P : |T|=k+1\}} S_T(\mathbb{E})}_{\mathrm{C}^k(\mathbb{E})} \to \cdots \to \underbrace{S_P(\mathbb{E})}_{\mathrm{C}^{N-1}(\mathbb{E})} \to 0 \to \cdots$$

where $S_T(\mathbb{E})$ is a special subspace of $\bigotimes_{t \in T} \mathcal{A}_t$ determined by $\mathbb{E}$. An element $r \in \mathrm{C}^k(\mathbb{E})$ can be thought of as an assignment that takes in any subset $T \subseteq P$ of size $|T| = k+1$ and outputs a random variable/operator $r_T \in \bigotimes_{t \in T} \mathcal{A}_t$ (living in a special subspace).[4] As suggested via the diagram above, representatives of the cohomology of $\mathrm{C}(\mathbb{E})$ are assignments (or tuples) of random variables that exhibit non-local correlations, acting as obstructions to the factorization of the expectation value. In the purely quantum situation the existence of such tuples of random variables is a phenomenon related to entangled states, and made famous by accounts of the Einstein-Podolsky-Rosen paradox.

Moreover, these cochain complexes can be thought of as a step toward a categorification of *multivariate mutual informations*: $\mathbb{R}$-valued quantities defined as an alternating sum of entropies that quantify the information shared among all subsystems [11]. The multivariate mutual information associated to a multipartite expectation value vanishes if the expectation value is factorizable, making it a good indicator of factorizability. Bivariate mutual information is particularly famous among high energy theorists, having important finiteness properties for field-theoretic states [12]. Given a multipartite expectation value, I define an associated three-parameter invariant[5]—called the *state index*—which behaves like an "Euler characteristic" in the sense that it behaves well under tensor products of expectation values and a certain notion of "gluing" of expectation values. Moreover, the state index is holomorphic in each of its parameters; by taking certain limits we see that it

[3] When working with infinite dimensional W^*-algebras, we work with Banach spaces.

[4] The differential is constructed by "lifting" such assignments to subsets of size $(k+1)$ to subsets of size $(k+2)$ roughly by "tensoring by the identity". More formally one constructs complexes by noticing the GNS construction helps forms a presheaf over the (discrete) space of tensor factors, and then taking associated Čech complexes.

[5] Here "invariant" means that this function is invariant under automorphisms of random variables that decompose as a tensor product of automorphisms on each tensor factor.

interpolates between multipartite mutual information, Tsallis/Renyí deformations [13, 14, 15, 16] of mutual information [1, §8.5], and the integer-valued Euler characteristics of our complexes.

In a more sophisticated treatment, one should be able to recover the full three-parameter state index from cohomological techniques. But even for the complexes of vector spaces developed so far, cohomology gives more than a numerical measure of how much information is shared between subsystems: elements of cohomology specify *what* information is shared among subsystems in the form of tuples of random variables. Moreover, there are entangled states for which multivariate mutual information vanishes, but have non-vanishing cohomology (see §2.3). This is analogous to how the Euler characteristic of any orientable compact manifold of odd-dimension is vanishing, making it useless for detecting the difference between two such manifolds (as opposed to the calculation of homology).

This work is closely related to that of Baudot and Bennequin [17] (J.P. Vigneaux provides an excellent exposition in [18]), who independently construct chain complexes of functions on spaces of probability measures such that mutual informations (and their Tsallis q-deformations) arise as generators of the first cohomology group. My work differs from theirs in the sense that I have a cohomology theory associated to a fixed measure (rather than the space of measures), allowing the mutual informations to arise as Euler characteristics rather than cocycles; however, the two theories are undoubtedly intimately related. I am hoping to elucidate this relationship, perhaps by carefully phrasing their construction in the language of obstruction theory and classifying stacks.

2.1. **Software.** As a hands-on supplement to [1], I developed software for calculating the cohomologies of multipartite quantum states, which is available on GitHub.[6] At the simplest level, the software takes in an N-partite (density) state and outputs associated degree $(N-1)$ Poincaré polynomials; however, it can also be used to compute explicit generators for cohomology.

2.2. **Goals and Future Work.** A few interrelated goals of this research are as follows:

(1) New invariants constructed from topological field theories: e.g. link invariants in Chern Simons theory.
(2) Applications to the study of symmetry protected topological phases.
(3) The development of measures of shared information both quantum mechanically, and classically: the quantum mechanical case being relevant to entanglement in quantum information theory and the latter possibly relevant to the analysis of large data sets;
(4) An understanding of the role of homotopy theory/homological algebra in *non-commutative probability theory*: the study of expectation value functions on (possibly) non-commutative algebras;
(5) Developing rigorous techniques for exploring the connection between entanglement and non-trivial geometries or topologies (e.g. the "gauge/gravity" or "AdS/CFT" correspondence).

With regard to (1): One can use Chern Simons theory [19, 20] to assign N-partite state to complements of (framed) N-component links in the three sphere.[7] By taking Poincaré polynomials of the cohomologies associated to such multipartite states, one produces link invariants. The relation of these link invariants to known link invariants remains unknown.

With regard to (2): There is a generalization of these homological techniques to the situation where one has an action of some symmetry group G on each tensor factor; in particular, there is a generalization of the state index to such a G-equivariant situation—giving an interesting G-equivariant generalization of mutual information/entanglement entropy that might be suitable to the study of symmetry protected topological phases of matter. I am currently exploring such

[6] See: https://github.com/tmainiero/homological-tools-4QM-octave and https://github.com/tmainiero/homological-tools-4QM-mathematica

[7] Such states are studied in [21, 22, 23].

applications in the context of matrix product states (which model the ground states of gapped Hamiltonians on one-dimensional lattices [24]); this research also has deep ties to (1) (see: [25, 26]).

With regard to (4): the machinery behind these complexes suggests that multipartite states behave like spaces, and there is a whole homotopy theory associated to them. Indeed, to every multipartite expectation value, there is a geometric object: formally given by a simplicial object in an appropriate "category of states". Our cohomological techniques are simple probes of this geometric object.

Moreover, there are certainly connections with the work of Drummond-Cole, Park, and Terilla [27, 28, 29, 30] who approach non-commutative probability theory from an A_∞/L_∞-perspective. One sophisticated version of the chain complexes above admits the structure of a differential graded module for a differential graded algebra; the resulting cohomology is an A_∞-module of some A_∞-algebra. Very little is known about these higher algebraic structures at the moment but, drawing vague analogies with the way that Massey products can identify subtle linkage properties of knot complements in the three sphere, my hope is that the higher structures can detect the Borromean-like entanglement properties of the tripartite Greenberger-Horne-Zeilinger (GHZ) state [31, 32].

I also believe that a homological/homotopical approach to (non-commutative) probability theory might lead to a geometrically inspired proof of strong subadditivity: a relation between entropies on a tripartite system, expressible as a simple inequality between bipartite mutual informations. Strong subaddivity of quantum entropies is a non-trivial statement, first proven by Leib and Ruskai [33]. However, Ryu-Takayanagi formulae lead to a very simple geometric interpretation of quantum strong subadditivity; the caveat is that these formulae only hold for small class of states that have special field theoretic descriptions. The homological/homotopical techniques mentioned above are not bound to this limitation.

With regard to (5): as discussed above, there is a deep duality between the classical geometry of general relativity and entangled states: e.g. the Ryu-Takayanagi formula, or even more vague statements such as EPR (Einstein-Podolsky-Rosen) = ER (Einstein-Rosen) [34]. Currently, numerical quantities such as mutual informations or entanglement entropies are used to study this duality; however, by their very nature as methods for studying space, cohomological techniques are likely to shed more light. Furthermore, it would be interesting if there were a connection with other perspectives on entanglement which draw on ideas of the linkage between entanglement and topology by associating entangled states to braids [35].

2.3. **A Sample of Results.** As a small taste of the cohomological techniques above, we begin by providing a theorem for bipartite pure states. In the following: if $\mathcal{H}$ is a Hilbert space $\mathrm{End}(\mathcal{H})$ denotes its space of (bounded) endomorphisms.

Theorem. *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces and $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$. Then there is a chain complex $\mathrm{G}(\psi)$ of $\mathbb{C}$-vector spaces (concentrated in degrees $-1, 0$ and 1) such that*

1. *The zeroth cohomology component $H^0[\mathrm{G}(\psi)]$ is naturally identifiable with a quotient of the space of pairs $(a, b) \in \mathrm{End}(\mathcal{H}_A) \times \mathrm{End}(\mathcal{H}_B)$ that are "maximally correlated" in a precise sense (in particular, $\langle \psi, (a \otimes 1_B - 1_A \otimes b)^*(a \otimes 1_B - 1_A \otimes b)\psi \rangle = 0$);*
2. $\dim_{\mathbb{C}} H^0[\mathrm{G}(\psi)] = r^2 - 1$*, where r is the "Schmidt rank" of ψ; in particular ψ is entangled if and only if $H^0 \not\equiv 0$. Moreover, one can use the Schmidt decomposition of ψ to generate a basis for H^0.*

As an example, if $\mathcal{H}_X$ is the orthonormal $\mathbb{C}$-span of e_X and f_X for $X \in \{A, B\}$, and we take ψ to be the Bell-state $\frac{1}{\sqrt{2}}(e_A \otimes e_B + f_A \otimes f_B)$, then

$$H^0[\mathrm{G}(\psi)] \cong \mathrm{span}_{\mathbb{C}} \left\{ (x_A \otimes y_A^\vee, x_B \otimes y_B^\vee) : x, y \in \{e, f\} \right\} / \mathrm{span}_{\mathbb{C}} \left\{ (1_A, 1_B) \right\};$$

where 1_X denotes the identity operator on $\mathcal{H}_X$, and $x_X^\vee := \langle x, - \rangle_{\mathcal{H}_X}$. This can be thought of as a quotient of the vector space of all non-locally correlated pairs of operators ("random variables") by the one-dimensional subspace spanned by the "trivial" correlated operators (i.e. identity operators).

As an example of what multipartite versions of our complexes can accomplish, we state some results relevant to the tripartite GHZ state and the W-state:

$$\mathrm{GHZ} := \frac{1}{\sqrt{2}} (e \otimes e \otimes e + f \otimes f \otimes f) \in (\mathrm{span}_{\mathbb{C}}\{e, f\})^{\otimes 3},$$
$$\mathrm{W} := \frac{1}{\sqrt{3}} (f \otimes e \otimes e + e \otimes f \otimes e + e \otimes e \otimes f) \in (\mathrm{span}_{\mathbb{C}}\{e, f\})^{\otimes 3}.$$

Both states are entangled and are known to be in distinct *SLOCC classes*, i.e. equivalence classes under the action of "local automorphisms": transformations of the form $A_1 \otimes A_2 \otimes A_3 \in \mathrm{GL}_2(\mathbb{C})^{\otimes 3}$ [36]. The tripartite mutual information is invariant under such transformations, and can be thought of as an indicator of information shared among all three tensor factors when non-vanishing. However, it vanishes on both of these states: failing to distinguish their SLOCC equivalence classes, detect entanglement, or detect shared information among all three tensor factors.

Poincaré polynomials associated to cohomologies of tripartite states are also invariant under local automorphisms. Recall that the Poincaré polynomial associated to a cochain complex C is given by:

$$P[\mathrm{C}] := \sum_{k \in \mathbb{Z}} y^k \dim_{\mathbb{C}} H^k[\mathrm{C}] \in \mathbb{Z}[y].$$

For a state ψ on a tensor product of Hilbert spaces, we can associate two cochain complexes: the *GNS complex* $\mathrm{G}(\psi)$ (constructed from the Čech complex associated to a presheaf of GNS representations) and the *commutant complex* $\mathrm{E}(\psi)$ (constructed from the Čech complex associated to a presheaf of commutants of GNS representations). The Poincaré polynomials of the GNS complexes of our tripartite states are:

$$P[\mathrm{G}(\mathrm{GHZ}_3)] = P[\mathrm{G}(\mathrm{W}_3)] = 1 + 6y.$$

which do not distinguish SLOCC classes. However, the fact that these polynomials are non-vanishing tells us that there exist tuples of operators that exhibit non-local correlations; something the tripartite mutual information fails to detect. In particular there is a single tuple of 1-body operators exhibiting non-trivial correlations between individual tensor factors, and six linearly independent tuples of 2-body operators that exhibit non-trivial correlations between pairs of tensor factors. Explicit generators for the associated cohomology components can be found in [1].

The resulting Poincaré polynomials for the commutant complexes are:

$$P[\mathrm{E}(\mathrm{GHZ})] = 7 + 7y,$$
$$P[\mathrm{E}(\mathrm{W})] = 3 + 3y.$$

As a result, our Poincaré polynomials detect the information shared among tensor factors due to entanglement, and distinguish the SLOCC classes of the W and GHZ state.

3. DT-invariants and Spectral Networks

Donaldson-Thomas (DT) Invariants [37, 38, 39, 40, 41] are (virtual) counts of (semi-stable) objects in a 3-Calabi-Yau (CY) category C; specifically, DT-invariants arise as counts of:

(1) Special Lagrangians in some CY threefold ("A-branes"): $\mathsf{C} = \mathsf{Fuk}(X)$ for some CY threefold X;
(2) Coherent sheaves ("B-branes") on a CY threefold $X^\vee$: $\mathsf{C} = \mathsf{D^b Coh}(X^\vee)$ (the mirror-dual description to 1);

(3) Moduli of complex quiver representations (e.g. $\mathsf{C} = \mathsf{D}^{\mathsf{b}}\mathsf{Rep}_{\mathbb{C}}(Q)$ for some quiver Q without loops);

In physics, the quantities listed above are geometric manifestations of *BPS states*: states in a supersymmetric field theory that are, roughly-speaking, stable under small deformations of the parameters defining the theory [42]; sometimes equivalent (or dual) descriptions of the same theory allow all three geometric manifestations to play a role (e.g. Denef [43] describes a physical derivation of the relationship between special Lagrangians and quivers).

My thesis work was focused on calculating DT-invariants utilizing techniques developed by Gaiotto-Moore-Neitzke (GMN) [44, 45, 46, 47] in the physics literature—where DT-invariants are conjectured to coincide with "BPS-indices": a weighted-count of BPS states [48]. Specifically, the main theme of my thesis pertains to the algebraicity over $\mathbb{Q}(x)$—hereafter referred to just as *algebraicity*—of generating functions for DT-invariants. My main results are:

(1) explicit algebraic relations obeyed by DT-invariant generating functions;
(2) constraints on the asymptotics of DT-invariants/BPS-indices due to algebraicity.

Details can be found in [49] and my earlier work with Galakhov, Longhi, Moore, and Neitzke [50].

Algebraicity has implications for cluster varieties—varieties constructed by gluing together algebraic tori via certain birational maps: the generating functions for DT-invariants (associated to quivers) appear explicitly in such gluing transformations. Cluster varieties are themselves interesting objects: they sometimes arise as the total space of an integrable system (e.g. the Hitchin integrable system), and they offer explicit geometric realizations of mirror symmetry. In physics, the asymptotics induced by algebraicity imply that, in some supersymmetric field theories, the number of states of mass $\leq M$ grows exponentially with M. This indicates the presence of "maximum" temperatures (known as Hagedorn-temperatures) or possible phase transitions for such theories.

Kontsevich and Soibelman have an understanding of algebraicity for a large class of 3-CY categories[51]; however they claim that their proof uses rather indirect methods. Using machinery developed by GMN called *spectral networks* [47], one can see algebraicity directly, and algorithmically construct explicit algebraic relations obeyed by generating functions. A spectral network is a directed, decorated graph associated to a $\mathbb{Z}$-family of BPS states; algebraic relations for generating functions of the associated BPS-indices follows by a system of algebraic relations determined by the edges and vertices of this graph.

3.1. **Future Directions.** As noted, algebraicity has implications for those studying cluster-varieties. A related consequence is a need to carefully define GMN's non-abelianization map [47, §10]: a map that (conjecturally) supplies cluster coordinates to a moduli space of $\mathrm{GL}_K\mathbb{C}$-local systems (with flag data) on a fixed punctured (real, smooth) surface and equipped with fixed monodromy around punctures. This moduli space arises as a symplectic leaf of (one of) Fock and Goncharov's $\mathcal{X}$-space(s) [52]. I partially sketched a definition of a functorial version of the non-abelianization map where algebraicity plays a central role. I am also interested in a derivation of the spectral network formalism from a purely representation theoretic perspective: one can view the pushforward of local systems via a covering map as the process of applying the induced representation functor associated to a subgroup(oid) of the fundamental group(oid); the data of a spectral network allows one to define an induced representation-like functor in the situation of *branched* covers (which can be formulated in the language of groupoids).

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